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Volterra Integral Equations of First Kind by Using Anuj Transform

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ABSTRACT: Volterra Integral equations occurs in variety of branches like Physics, Chemistry, Biology and Engineering. These equations have been solved by using Laplace Transform, Sumudu Transform, Sawi Transform, Tarig Transform, Aboodh Transform and Kamal Transform. In this paper we use recently developed Anuj Transform for solving linear Volterra integral equations of first kind. Some problems are solved by using Anuj transform, this shows that Anuj transform is useful and effective for solving such equations.

Keywords: Integral transform, Anuj transform, Volterra integral equations of first kind, Convolution theorem.
AMS 2010: 45D05, 45E10, 65R10, 65R20.

I. INTRODUCTION:

Linear Volterra integral equation of first kind is of the form [1]

$$f(x) = \int_0^x K(x,t)\, u(t)dt \qquad (1.1)$$

Here $K(x,t)$ is Kernel, u is unknown function and $f(x)$ is real valued function. A linear Volterra integral equation is a convolution equation, if it has form

$$f(x) = \int_{t_0}^t K(t-s)\, u(s)ds \qquad (1.2)$$

Vito-Volterra introduced these equations in 1908, Further TraianLolescu studied these equations on his thesis Volterra integral equations has many applications in various fields like demography, study of Viscoelastic material and acturialscience through the renewal equations, are some of the fields. Nowadays, many researchers are engaged in introducing various types of integral transforms. Recently, in October 2021 S. S. Khakale and D. P. Patil [2] introduced Soham transform, which is defined for the function as

$$S[f(t)] = P(v) = \frac{1}{v}\int_0^\infty e^{-v^\alpha t} f(t)dt$$

Where α is non zero real number.

Anuj transform is introduced by Anuj Kumar, Shikha Bansal and Sudhanshu Aggarwal [3] in January 2022. For the function $f(t)$, Anuj transform is denoted and defined as

$$\Lambda\{f(t)\} = f(p) = p^2 \int_0^\infty e^{-\left(\frac{1}{p}\right)t} f(t)dt\,,\, p > 0 \qquad (1.3)$$

Where Λ denotes the Anuj transform operator

Recently in September 2021 Kushare and Patil [4] introduced Kushare transform, for simplifying the process of obtaining solution of ordinary and partial differential equations in the time domain, many researchers are interested to use newly developed integral transform in various fields. Some of them are as follows, Sudhanshu Aggarwal et al [5] used Kamal transform to solve linear Volterra integral equation of first kind. Recently, in January 2022 R. S. Sanap and D. P. Patil [6] obtain the solution of Newton’s Law of Cooling by using Kushare transform. In October 2021 Sawi transform used in Bessel function by D. P. Patil[7] further Sawi transform of Error function is used to evaluate improper integral by D. P. Patil [8]. Patil [9] used Laplace and Shehu transform in Chemical Sciences, Sawi transform and its convolution theorem is applied to solve wave equation by Patil [10]. Further, D. P. Patil [11] used Mahgoub transform for getting the solution OR Parabolic boundary value problems. Patil [12] also used double Laplace and double Sumudu transform to solve the wave equation. Dualities between various double integral transforms are obtained by D. P. Patil [13]. Laplace, Elzaki and Mahgoub transforms are used for solving system of first order and first-degree differential equations by Kushare and Patil [14]. Dr. Patil [15] also used Aboodh and Mahgoub transform in boundary value problems of system ordinary differential equations recently in April 2022, Nikam, Shirsath, Aher and Patil [16] used Kushare transform in growth and decay problems. Patil [17] compared Laplace, Sumudu, Aboodh, Elzaki and Mahgoub transform in evaluating boundary value problems. Further Patil [18] used double Mahgoub transform for solving parabolic boundary value problems.

In this paper we use Anuj transform for solving the linear Volterra integral equations of first kind. Paper is organized as follows, in second section definition of Anuj transform is given. Anuj transform of some functions are included in third section. Fourth section is devoted for the properties of Anuj transform of derivative. fifth section is for Anuj transform of

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derivative of functions. Inverse of Anuj transform is explained in sixth section. In seventh section Anuj transform is applied to linear Volterra integral equations of first Kind. Some applications are in eighth section and conclusion is drawn in last section.

II. DEFINATION OF ANUJ TRANSFORM [3]

The Anuj transform of a piecewise continuous exponential order function F(t),
$t \geq 0$ is given by

$$\Lambda\{F(t)\} = p^2 \int_0^\infty F(t) e^{-\left(\frac{1}{p}\right)t}\, dt = f(p), \quad p > 0 \qquad (2.1)$$

Here Λ denotes the Anuj transform operator.

III. ANUJ TRANSFORM OF FREQUENTLY USED FUNCTIONS

S.N.	F(t)	$\Lambda\{F(t)\} = f(p)$
1.	1	p^3
2.	t	p^4
3.	t^2	$2p^5$
4.	$t^n, n \in N$	$n!\,p^{n+3}$
5.	$t^n, n > -1$	$p^{n+3}\Gamma(n+1)$
6.	e^{at}	$\frac{p^3}{1-ap}$
7.	$sinat$	$\frac{ap^4}{1+a^2p^2}$
8.	$cosat$	$\frac{p^3}{1+a^2p^2}$
9.	$sinhat$	$\frac{ap^4}{1-a^2p^2}$
10.	$coshat$	$\frac{p^3}{1-a^2p^2}$

IV. SOME OPERATIONAL PROPERTIES OF ANUJ TRANSFORM

In this section we state some properties of Anuj transform.

4.1 Linearity Property of Anuj Transform

If Anuj transform of functions $F_1(t)$ and $F_2(t)$ are $f_1(p)$ and $f_2(p)$ respectively then Anuj transform of $[lF_1(t) + mF_2(t)]$ is given by $l f_1(p) + m f_2(p)$, where l, m are arbitrary constants.

4.2 Scale Property of Anuj Transform

If $\Lambda\{F(t)\} = f(p)$ then $\Lambda\{F(kt)\} = \frac{1}{k^3} f(kp)$.

4.3 Translation Property of Anuj Transform

If $\Lambda\{F(t)\} = f(p)$ then $\Lambda\{e^{kt}\,F(t)\} = (1-kp)^2 f\left(\frac{p}{1-kp}\right)$.

4.4 Convolution Property of Anuj Transform

If $\Lambda\{F_1(t)\} = f_1(p)$ and $\Lambda\{F_2(t)\} = f_2(p)$ then

$\Lambda\{F_1(t) * F_2(t)\} = \frac{1}{p^2}\Lambda\{F_1(t)\}\Lambda\{F_2(t)\} = \frac{1}{p^2} f_1(p) f_2(p)$, where convolution of $F_1(t)$ and $F_2(t)$ is denoted by $F_1(t) * F_2(t)$ and it is defined by $F_1(t) * F_2(t) = \int_0^t F_1(t-u)F_2(u)du = \int_0^t F_1(u)F_2(t-u)du$.

V. ANUJ TRANSFORM OF DERIVATIVES OF FUNCTION

If $\Lambda\{F(t)\} = f(p)$ then

a) $\Lambda\{F'(t)\} = \frac{1}{p} f(p) - p^2 F(0)$

b) $\Lambda\{F''(t)\} = \frac{1}{p^2} f(p) - pF(0) - p^2 F'(0)$

c) $\Lambda\{F'''(t)\} = \frac{1}{p^3} f(p) - F(0) - pF'(0) - p^2 F''(0)$

VI. INVERSE ANUJ TRANSFORM

The inverse Anujtransform of $f(p)$, designated by $\Lambda^{-1}\{f(p)\}$, is another function $F(t)$ having the property that $\Lambda\{F(t)\} = f(p)$. Here Λ^{-1} denotes the inverse Anuj transform operator.

6.1 INVERSE ANUJ TRANSFORM OF FREQUENTLY USED FUNCTIONS

S.N.	$f(p)$	$F(t) = \Lambda^{-1}f(p)$
1.	p^3	1
2.	p^4	t
3.	p^5	$\frac{t^2}{2!}$
4.	$p^{n+3}, n \in N$	$\frac{t^n}{n!}$
5.	$p^{n+3}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6.	$\frac{p^3}{1-ap}$	e^{at}
7.	$\frac{p^4}{1+a^2p^2}$	$\frac{sinat}{a}$
8.	$\frac{p^3}{1+a^2p^2}$	$cosat$
9.	$\frac{p^4}{1-a^2p^2}$	$\frac{sinhat}{a}$
10.	$\frac{p^3}{1-a^2p^2}$	$coshat$

6.2. LINEARITY PROPERTY OF INVERSE ANUJ TRANSFORM

If $\Lambda^{-1}\{f_1(p)\} = F_1(t)$ and $\Lambda^{-1}\{f_2(p)\} = F_2(t)$ then

$$\Lambda^{-1}\{l\, f_1(p) + m f_2(p)\} = \Lambda^{-1} l\{f_1(p)\} + m\Lambda^{-1}\{f_2(p)\}$$

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$\Rightarrow \Lambda^{-1}\{l\, f_1(p) + m\, (p)\} = l\, F_1(t) + m\, F_2(t)$, where l, m are arbitrary constants.

In general, $\Lambda^{-1}\{\sum_{i=1}^{n} l_i f_i(p)\} = \sum_{i=1}^{n} l_i \Lambda^{-1}\{f_i(p)\}$, where l_i are arbitrary constants.

VII. ANUJ TRANSFORM FOR SOLVING LINEAR VOLTERRA INTEGRALEQUATION OF FIRST KIND WITH CONVOLUTION TYPE KERNEL

The linear Volterra integral equation of first kind with convolution type Kernel is given by

$\theta(t) = \int_0^t K(t-u)\mu(u)du$(7.1)

Where

$\mu(t)$is *unknown function*,$\theta(t)$*is known function* and $K(t-u)$is *convolution type kernel*

Operating Anuj Transform on equation (7.1), we get

$$\Lambda\{\theta(t)\} = \Lambda\{\int_0^t K(t-u)\,\mu(u)du\}$$

$\Rightarrow \Lambda\{\theta(t)\} = \Lambda\{K(t) * \mu(t)\}$ (7.2)

Using convolution theorem in equation (7.2), we have

$$\Lambda\{\theta(t)\} = \frac{1}{p^2}\Lambda\{K(t)\}\Lambda\{\mu(t)\}$$

$\Lambda\{\mu(t)\} = p^2\left(\frac{\Lambda\{\theta(t)\}}{\Lambda\{K(t)\}}\right)$(7.3)

After operating inverse Anuj transform on equation (7.3), the required solution of equation (7.1) obtain and it is given by

$$\mu(t) = \Lambda^{-1}\left\{p^2\left(\frac{\Lambda\{\theta(t)\}}{\Lambda\{K(t)\}}\right)\right\}$$

VIII. APPLICATIONS:

In this section, we give some applicationsof Anuj transform in Volterra integral equations of first kind.This demonstrates the effectiveness of Anuj transform for solving linear Volterra integral equation of first kind.

Application:1

Consider the following linear Volterra integral equation of first kind with convolution type kernel

$$sint = \int_0^t e^{(t-u)}\mu(u)du \quad (8.1)$$

Operating Anuj transform on equation (8.1), we get

$$?\{sint\} = \Lambda\left\{\int_0^t e^{(t-u)}\mu(u)du\right\}$$

$$\Rightarrow \frac{p^4}{1+p^2} = \Lambda\{e^t * \mu(t)\} \quad (8.2)$$

Using convolution theorem in equation (8.2), we have

$$\frac{p^4}{1+p^2} = \frac{1}{p^2}\Lambda\{e^t\}?\{\mu(t)\}$$

$$\Rightarrow \frac{p^4}{1+p^2} = \frac{1}{p^2}\left(\frac{p^3}{1-p}\right)\Lambda\{\mu(t)\}$$

$$\Rightarrow \frac{p^4}{1+p^2} = \left(\frac{p}{1-p}\right)\Lambda\{\mu(t)\}$$

$$\Rightarrow \Lambda\{\mu(t)\} = \frac{p^4}{1+p^2}\left(\frac{1-p}{p}\right)$$

$$\Rightarrow \Lambda\{\mu(t)\} = \frac{p^3(1-p)}{1+p^2}$$

$$\Rightarrow \Lambda\{\mu(t)\} = \frac{p^3}{1+p^2} - \frac{p^4}{1+p^2} \quad (8.3)$$

After operating inverse Anuj transform on equation (8.3), the required solution of equation (8.1) obtain and it is given by

$$\mu(t) = \Lambda^{-1}\left\{\frac{p^3}{1+p^2} - \frac{p^4}{1+p^2}\right\}$$

$$\Rightarrow \mu(t) = \Lambda^{-1}\left\{\frac{p^3}{1+p^2}\right\} - \Lambda^{-1}\left\{\frac{p^4}{1+p^2}\right\}$$

$$\mu(t) = cost - sint$$

Application:2

Consider the following linear Volterra integral equation of first kind with convolution type kernel

$$t^2 = \int_0^t (t-u)\,\mu(u)\,du \quad (8.4)$$

Operating Anuj transform on equation (8.4), weget

$$\Lambda\{t^2\} = \frac{1}{2}\Lambda\left\{\int_0^t (t-u)\,\mu(u)\,du\right\}$$

$$2p^5 = \frac{1}{2}\Lambda\{t * \mu(t)\} \quad (8.5)$$

Using convolution theorem in equation (8.5), we get

$$4p^5 = \frac{1}{p^2}\Lambda\{t\}\,\Lambda\{\mu(t)\}$$

$$\Rightarrow 4p^5 = \frac{1}{p^2}p^4\,\Lambda\{\mu(t)\}$$

$$\Rightarrow 4p^5 = p^2\,\Lambda\{\mu(t)\}$$

$\Rightarrow \Lambda\{\mu(t)\} = 4p^3$(8.6)

After operating inverse Anuj transform on equation (8.6), the required solution of equation (8.4) obtain and it is given by

$$\mu(t) = 4\Lambda^{-1}\{p^3\}$$
$$\mu(t) = 4(1)$$
$$\mu(t) = 4$$

Application:3

Consider the following linear Volterra integral equation of first kind with convolution type kernel

$$t = \int_0^t e^{-(t-u)}\,\mu(u)\,du \quad (8.7)$$

Operating Anuj transform on equation (8.7), we get

$$\Lambda\{t\} = \Lambda\left\{\int_0^t e^{-(t-u)}\,\mu(u)du\right\}$$

$$\Rightarrow p^4 = \Lambda\{e^{-t} * \mu(t)\} \quad (8.8)$$

Using convolution theorem in equation (8.8), we have

$$p^4 = \frac{1}{p^2}\Lambda\{e^{-t}\}\,\Lambda\{\mu(t)\}$$

$$\Rightarrow p^4 = \frac{1}{p^2}\left(\frac{p^3}{1+p}\right)\Lambda\{\mu(t)\}$$

$$\Rightarrow p^4 = \left(\frac{p}{1+p}\right)\Lambda\{\mu(t)\}$$

$$\Rightarrow \Lambda\{\mu\} = \frac{p^4(1+p)}{p}$$

$$\Rightarrow \Lambda\{\mu\} = p^3(1+p)$$

$$\Rightarrow \Lambda\{\mu\} = p^3 + p^4 (8.9)$$

After operating inverse Anuj transform on equation (8.9), the required solution of equation (8.7) obtain and it is given by

$$\mu(t) = \Lambda^{-1}\{p^3 + p^4\}$$

$$\Rightarrow \mu(t) = \Lambda^{-1}\{p^3\} + \Lambda^{-1}\{p^4\}$$

$$\Rightarrow \mu(t) = 1 + t$$

Application:4

Consider the following linear Volterra integral equation of first kind with convolution type kernel

$$t = \int_0^t \mu(u)\,du \quad (8.10)$$

Operating Anuj transform on equation (8.10), we get

$$\Lambda\{t\} = \Lambda\left\{\int_0^t \mu(u)\,du\right\}$$

$$\Rightarrow p^4 = \Lambda\{1 * \mu(t)\} (8.11)$$

Using convolution theorem in equation (8.11), we have

$$p^4 = \frac{1}{p^2}\Lambda\{1\}\,\Lambda\{\mu(t)\}$$

$$\Rightarrow p^4 = \frac{1}{p^2}(p^3)\,\Lambda\{\mu(t)\}$$

$$\Rightarrow p^4 = p\,\Lambda\{\mu(t)\}$$

$$\Rightarrow \Lambda\{\mu(t)\} = \frac{p^4}{p}$$

$$\Rightarrow \Lambda\{\mu(t)\} = p^3 (8.12)$$

After operating inverse Anuj transform on equation (8.12), the required solution of equation (8.10) obtain and it is given by

$$\mu(t) = \Lambda^{-1}\{p^3\}$$

$$\Rightarrow \mu(t) = 1$$

Conclusion: We successfully used Anuj transform for solving linear Volterra integral equations of first kind.

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