International Journal of Advances in Engineering and Management (IJAEM)
Volume 4, Issue 5 May 2022, pp: 2896-2902www.ijaem.net ISSN: 2395-5252

Teaching ofconstant variation methodof constant ordinary differential equation class and its expansion in higher order equation

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Submitted: 20-05-2022 Revised: 30-05-2022 Accepted: 02-06-2022

ABSTRACT: Ordinary differential equations play an important role in the study of basic mathematics. In solving this kind of equations, constant variation method is one of the most commonly used methods. In this paper, the expansion of constant variation method in solving several kinds of high-order differential equations is summarized, and specific application examples are given to help the learners study ordinary differential course.

Key words: Constantvariationmethod; higher order differential equation; general solution; special solution

I. INTRODUCTION

Equation is an important part of mathematics. In the previous study of equations, we know that equations are connected by known numbers and unknown numbers [1–3] . In some practical problems, establishing equations can simplify the problem and make the research more convenient. However, with the deepening of study, we found that when studying some physical models, we need to use quantities that can accurately reflect the motion law of objects, and these quantities are usually expressed as the derivatives of some quantities in mathematics. After the mathematical equations of these physical models are constructed, the equations need to be solved. When solving such equations, constant variation method is one of the most commonly used methods[4–5] . Therefore, it is necessary to study the solution of ordinary differential equations by Constant variation method.

Constant variation method is a method for finding the general solution of ordinary differential equations developed by French mathematician Lagrange after 11 years of research[6] . This method solves the problem of finding the general solution of the first order linear non-homogeneous differential equation, and provides a theoretical basis for the study of other kinds of differential equations. Its main idea is: first find the general solution of the non-homogeneous linear differential equation corresponding to the homogeneous equation, replace the constant coefficient in the general solution with the undetermined function, assume that the replaced formula is the solution of the original equation, and then replace it back to the original equation to find the analytical formula of the undetermined function, so as to obtain the general solution of this kind of non-homogeneous linear differential equation[7–8] .The essence of this method is a special variable substitution method, which can be extended to higher-order differential equations and systems of differential equations.

II. MAIN RESULTS

2.1Application of constant variation method to second order linear differential equations with constant coefficients

Consider the following second order nonhomogeneous linear differential equations with constant coefficients

$$y'' + py' + qy = f(x) \quad (1)$$

The corresponding homogeneous equation is

$$y'' + py' + qy = 0 \quad (2)$$

Its characteristic equation is

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Volume 4, Issue 5 May 2022, pp: 2896-2902www.ijaem.net ISSN: 2395-5252

$$r^2 + pr + q = 0 \qquad (3)$$

If the two linearly independent solutions of equation (2) are known to be $x_1(t)$ and $x_2(t)$, the general solution of equation (2) can be expressed as $y = c_1 x_1(t) + c_2 x_2(t)$. By usingconstant variation method, let the general solution of equation (1) be $y = c_1(t)x_1(t) + c_2(t)x_2(t)$, substitute it into the original equation to obtain $c_1(t), c_2(t)$, and then obtain the general solution.**Remark1.**If a is the real root of equation (3), then $y = ce^{ax}$ is the solution of equation (2). Let a special solution of equation (1) be $y = c(x)e^{ax}$, and substitute it into the original equation to obtain

$$c''(x) + (2a + p)c'(x) = e^{-ax} f(x)$$

By solving the above equation

$$c(x) = \int \left[e^{-(2a+p)x} \left(\int e^{(a+p)x} f(x) dx \right) \right] dx$$

Then we obtain

$$y = e^{ax} \int \left[e^{-(2a+p)x} \left(\int e^{(a+p)x} f(x) dx \right) \right] dx$$

Remark2. If a is the complex root of equation (3), it is advisable to set $a = m + ni$ $(m, n \in R, n \neq 0)$,so $y = e^{mx} \sin nx$ is the solution of equation (2). According to constant variation method, a special solution of equation (1) can be set as $y = c(x)e^{mx} \sin nx$, substituted into the original equation to calculate $c(x)$, and then a special solution of equation (1) can be obtained as

$$y = e^{mx} \sin nx \times \int \frac{e^{-(2m+p)x} \left(\int e^{(m+p)x} f(x) \sin nx dx \right)}{\sin^2 nx} dx$$

2.2Application of constant variation method to second order linear differential equations with variable coefficients

Consider a second order nonhomogeneous linear differential equation with variable coefficients

$$y'' + p(x)y' + q(x)y = f(x) \qquad (4)$$

If equation (4) has a special solution y_1, by the method of variation of constant, let $y^* = c(x)y_1$,then

$$y^{*\prime} = c'(x)y_1 + c(x)y_1',$$

$$y^{*\prime\prime} = c''(x)y_1 + c'(x)y_1' + c'(x)y_1' + c(x)y_1''$$

Substituting $y^{*\prime}, y^{*\prime\prime}$ into equation $y'' + p(x)y' + q(x) = f(x)$ yields

$$c''(x)y_1 + [p(x)y_1 + 2y_1']c'(x)$$
$$+ [y_1'' + p(x)y_1' + q(x)y_1]c(x) = f(x)$$

It can be seen from section 2.1

$$y_1'' + p(x)y_1' + q(x)y_1 = 0$$

therefore

$$c''(x)y_1 + [p(x)y_1 + 2y_1']c'(x) = f(x)$$

Divide both sides of the equation by y_1

$$c''(x) + [p(x) + \frac{2y_1'}{y_1}]c'(x) = \frac{f(x)}{y_1}$$

Then $c'(x)$ can be calculated according to the general solution formula

$$c(x) = \int \left[\frac{e^{-\int p(x)dx}}{y_1^2} \int y_1 f(x) e^{\int p(x)dx} dx \right] dx$$

i.e.

$$y^* = y_1 \int \left[\frac{e^{-\int p(x)dx}}{y_1^2} \int y_1 f(x) e^{\int p(x)dx} dx \right] dx$$

Thus, the general solution is

$$y = c_1 y_1 + c_2 y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$$
$$+ y_1 \int \left[\frac{e^{-\int p(x)dx}}{y_1^2} \int y_1 f(x) e^{\int p(x)dx} dx \right] dx$$

2.3Application of constant variation method in n-order nonhomogeneous linear differential equations

Consider the following n -order nonhomogeneous linear differential equations

$$\frac{d^n x}{dt^n} + a_1(t) \frac{d^{(n-1)} x}{dt^{(n-1)}} + \cdots + a_n(t)x = f(t) \quad (5)$$

where $a_i(t)$ $(i = 1, 2, \cdots, n)$ and $f(t)$ are continuous functions on $a \le t \le b$.

If $f(t) \equiv 0$, the n -order homogeneous linear differential equation corresponding to equation (5) is obtained

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Volume 4, Issue 5 May 2022, pp: 2896-2902www.ijaem.net ISSN: 2395-5252

$$\frac{d^n x}{dt^n}+a_1(t)\frac{d^{(n-1)}x}{dt^{(n-1)}}+\cdots+a_n(t)x=0 \quad (6)$$

Definition1Error! Reference source not found. The determinants of k differentiable $k-1$ degree functions $x_1(t),x_2(t),\cdots,x_k(t)$ defined on interval $a\le t\le b$ are called the Wronskian determinants of these functions such as

$$W(t)\equiv W[x_1(t),x_2(t),\cdots,x_k(t)]$$
$$\equiv\begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_k(t)\\ x_1'(t) & x_2'(t) & \cdots & x_k'(t)\\ \vdots & \vdots & \ddots & \vdots\\ x_1^{(k-1)}(t) & x_2^{(k-1)}(t) & \cdots & x_k^{(k-1)}(t)\end{vmatrix}$$

Let $x_1(t),x_2(t),\cdots,x_n(t)$ be n linearly independent solutions of equation (6), so the general solution of equation (6) is

$$x=c_1x_1(t)+c_2x_2(t)+\cdots+c_nx_n(t) \quad (7)$$

By using constant variation method, replace c_i $(i=1,2,\cdots,n)$ with $c_i(t)$ $(i=1,2\cdots n)$, then we have

$$x=c_1(t)x_1(t)+c_2(t)x_2(t)+\cdots+c_n(t)x_n(t) \quad (8)$$

By substituting it into equation (5), an equation satisfied by $c_1(t),c_2(t),\cdots,c_n(t)$ is obtained In order to find $c_1(t),c_2(t),\cdots,c_n(t)$, the other $n-1$ constraints need to be found. By differentiating t

$$x'=c_1(t)x_1'(t)+c_2(t)x_2'(t)+\cdots+c_n(t)x_n'(t)x$$
$$+c_1'(t)x_1(t)+c_2'(t)x_2(t)+\cdots+c_n'(t)x_n(t)$$

Let

$$c_1'(t)x_1(t)+c_2'(t)x_2(t)+\cdots+c_n'(t)x_n(t)=0$$

we obtain

$$x'=c_1(t)x_1'(t)+c_2(t)x_2'(t)+\cdots+c_n(t)x_n'(t)$$

therefore

$$x''=c_1(t)x_1''(t)+c_2(t)x_2''(t)+\cdots$$
$$+c_n(t)x_n''(t)x+c_1'(t)x_1'(t)$$
$$+c_2'(t)x_2'(t)+\cdots+c_n'(t)x_n'(t)$$

Let

$$c_1'(t)x_1'(t)+c_2'(t)x_2'(t)+\cdots$$
$$+c_n'(t)x_n'(t)=0$$

we can obtain

$$x''=c_1(t)x_1''(t)+c_2(t)x_2''(t)+\cdots$$
$$+c_n(t)x_n''(t)$$

Continue the above steps, and we get the $n-1$ condition

$$c_1'(t)x_1^{(n-2)}(t)+c_2'(t)x_2^{(n-2)}(t)+\cdots$$
$$+c_n'(t)x_n^{(n-2)}(t)=0$$

and

$$x^{(n-1)}=c_1(t)x_1^{(n-1)}(t)+c_2(t)x_2^{(n-1)}(t)+\cdots$$
$$+c_n(t)x_n^{(n-1)}(t)$$

Finally, by differentiating t, we can get

$$x^{(n)}=c_1(t)x_1^{(n)}(t)+c_2(t)x_2^{(n)}(t)+\cdots$$
$$+c_n(t)x_n^{(n)}(t)+c_1'(t)x_1^{(n-1)}(t)$$
$$+c_2'(t)x_2^{(n-1)}(t)+\cdots+c_n'(t)x_n^{(n-1)}(t)$$

Now we substitute $x',x'',\cdots,x^{(n)}$ into (17) and combine that $x_1(t),x_2(t),\cdots,x_n(t)$ is a linearly independent solution of equation (6), we can obtain

$$c_1'(t)x_1^{(n-1)}(t)+c_2'(t)x_2^{(n-1)}(t)+\cdots$$
$$+c_n'(t)x_n^{(n-1)}(t)=f(t)$$

Therefore we have

$$\begin{cases} c_1'(t)x_1(t)+c_2'(t)x_2(t)+\cdots+c_n'(t)x_n(t)=0\\ c_1'(t)x_1'(t)+c_2'(t)x_2'(t)+\cdots+c_n'(t)x_n'(t)=0\\ \cdots\\ c_1'(t)x_1^{(n-1)}(t)+c_2'(t)x_2^{(n-1)}(t)+\cdots\\ +c_n'(t)x_n^{(n-1)}(t)=f(t)\end{cases}$$

The coefficient determinant of this system of equations $W(t)\not\equiv 0$, that is, the equations has only a unique solution, which can be obtained by letting $c_i'(t)=\varphi_i(t)$ $(i=1,2,\cdots,n)$

$$c_i(t)=\int\varphi_i(t)dt+\gamma_i \qquad \forall\gamma_i\in R,$$
$$i=1,2,\cdots,n$$

Substitute $c_i(t)$ into (8) to obtain the solution of equation (5) as

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$$x=\sum_{i=1}^{n}\gamma_i x_i(t)+\sum_{i=1}^{n}x_i(t)\int\varphi_i(t)dt$$

where γ_i $(i=1,2,\cdots,n)$ are constants.

2.4 Application of constant variation method to linear differential equations

Consider nonhomogeneous linear differential equations

$$\boldsymbol{x}'=\boldsymbol{A}(t)\boldsymbol{x}+\boldsymbol{f}(t) \quad (9)$$

where $\boldsymbol{A}(t)$ is a known continuous matrix on interval $a\le t\le b$, and $\boldsymbol{f}(t)$ is a known dimensional continuous column vector on interval $a\le t\le b$.

When $\boldsymbol{f}(t)=0$, the homogeneous linear differential equations corresponding to equation (9) are obtained

$$\boldsymbol{x}'=\boldsymbol{A}(t)\boldsymbol{x} \quad (10)$$

If $\boldsymbol{\phi}(t)$ is the solution of (10), let $\boldsymbol{\varphi}(t)=\boldsymbol{\phi}(t)\boldsymbol{c}(t)$ be the solution of (9) and substitute it into (9) by using constant variation method, we obtain

$$\boldsymbol{\phi}(t)\boldsymbol{c}'(t)+\boldsymbol{\phi}'(t)\boldsymbol{c}(t)=\boldsymbol{A}(t)\boldsymbol{\phi}(t)\boldsymbol{c}(t)+\boldsymbol{f}(t)$$

With $\boldsymbol{\phi}'(t)=\boldsymbol{A}(t)\boldsymbol{\phi}(t)$, we have

$$\boldsymbol{\phi}(t)\boldsymbol{c}'(t)=\boldsymbol{f}(t)$$

Multiply the left and right ends of the equation by $\boldsymbol{\phi}^{-1}(t)$, and then integrate to get

$$\boldsymbol{c}(t)=\int_{t_0}^{t}\boldsymbol{\phi}^{-1}(s)\boldsymbol{f}(s)ds,t_0,t\in[a,b]$$

where $\boldsymbol{c}(t_0)=0$. Therefore we get

$$\boldsymbol{\varphi}(t)=\boldsymbol{\phi}(t)\int_{t_0}^{t}\boldsymbol{\phi}^{-1}(s)\boldsymbol{f}(s)ds,t_0,t\in[a,b] \quad (11)$$

The formula (11) is a constant variation formula of non-homogeneous linear differential equations (9).

III. EXAMPLES

Example1 Given that a basic solution group of homogeneous equation corresponding to differential equation

$$t^2\frac{d^2x}{dt^2}-2t\frac{dx}{dt}+2x=2t^3$$

is $x_1=t,x_2=t^2$, find the general solution of the equation.

Solution: From the problem, we know that a basic solution group of the homogeneous equation corresponding to the original equation is $x_1=t,x_2=t^2$, we can obtain

$$W=\begin{vmatrix}x_1(t) & x_2(t)\\ x_1'(t) & x_2'(t)\end{vmatrix}=\begin{vmatrix}t & t^2\\ 1 & 2t\end{vmatrix}=t^2$$

$$W_1=\begin{vmatrix}0 & x_2(t)\\ \dfrac{f(t)}{a_0(t)} & x_2'(t)\end{vmatrix}=\begin{vmatrix}0 & t^2\\ 2t & 2t\end{vmatrix}=-2t^3$$

$$W_2=\begin{vmatrix}x_1(t) & 0\\ x_1'(t) & \dfrac{f(t)}{a_0(t)}\end{vmatrix}=\begin{vmatrix}t & 0\\ 1 & 2t\end{vmatrix}=2t^2$$

therefore,we obtain

$$\frac{W_1}{W}=-2t,\frac{W_2}{W}=2$$

Thus

$$\int\frac{W_1}{W}dt=\int-2tdt=-t^2$$

$$\int\frac{W_2}{W}dt=\int 2dt=2t$$

So the general solution of the original equation is

$$x(t)=\sum_{i=1}^{2}c_ix_i(t)+\sum_{i=1}^{2}x_i(t)\int\frac{W_i}{W}dt$$
$$=c_1t+c_2t^2-t\cdot t^2+t^2\cdot 2t$$
$$=c_1t+c_2t^2+t^3$$

where c_1,c_2 are constants.

Example2 Find the solution $\boldsymbol{\varphi}(t)$ of $\boldsymbol{x}'=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{f}(t)$ satisfying the initial value $\boldsymbol{\varphi}(0)=\begin{bmatrix}1\\-1\end{bmatrix}$, where

$$\boldsymbol{A}=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix},\boldsymbol{f}(t)=\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

Solution: With

$$\boldsymbol{\phi}'(t)=\begin{bmatrix}2e^{2t} & e^{2t}+2te^{2t}\\ 0 & 2e^{2t}\end{bmatrix}$$
$$=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}e^{2t} & te^{2t}\\ 0 & e^{2t}\end{bmatrix}$$
$$=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}\boldsymbol{\phi}(t)=\boldsymbol{A}\boldsymbol{\phi}(t)$$

and $|\boldsymbol{\phi}(t)| = e^{4t} \neq 0$, So

$$\boldsymbol{\phi}(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

is the basic solution matrix of $\boldsymbol{x}' = \boldsymbol{Ax}$.

By using constant variation method, let the solution is $\boldsymbol{\varphi}(t) = \boldsymbol{\phi}(t)\boldsymbol{c}(t)$ .And with

$$\boldsymbol{\phi}(t)\boldsymbol{c}'(t) = \boldsymbol{f}(t)$$

i.e.

$$\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} e^{2t}c_1'(t) + te^{2t}c_2'(t) \\ e^{2t}c_2'(t) \end{bmatrix} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

we can obtain

$$\begin{cases} c_1'(t) = e^{-2t}(\sin t - t\cos t) \\ c_2'(t) = e^{-2t}\cos t \end{cases}$$

So the nonhomogeneous equation has a solution

$$\boldsymbol{\varphi}(t) = \boldsymbol{\phi}(t)\boldsymbol{c}(t) = \begin{bmatrix} (c_1 + c_2 t)e^{2t} - \frac{2}{25}\cos t - \frac{14}{25}\sin t \\ c_2 e^{2t} + \frac{1}{5}\sin t - \frac{2}{5}\cos t \end{bmatrix}$$

where c_1, c_2 are constants.

When the initial value condition $\boldsymbol{\varphi}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $c_1 = \frac{27}{25}, c_2 = -\frac{3}{5}$ can be obtained, so there is a special solution

$$\boldsymbol{\varphi}(t) = \begin{bmatrix} \frac{1}{25}(-2\cos t - 14\sin t) + \frac{27}{25}e^{2t} - \frac{3}{5}te^{2t} \\ \frac{1}{5}\sin t - \frac{2}{5}\cos t - \frac{3}{5}e^{2t} \end{bmatrix}.$$

IV. CONCLUSION

Constant variation methodplays a very important role in the course of ordinary differential equations. It is very practical. In addition, when this method is used to solve high-order differential equations or equations, it broadens the idea of solving problems in a certain range. Although the amount of calculation is large, the theory is not difficult to understand, and it also plays a great role in improving our mathematical thinking ability.

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