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Some Existence Theorems for Functional Equation Arising In Two Person Zero-Sum Multistage Game.

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ABSTRACT

we have established the existence and uniqueness of the solutions for a general utility function. In our work we have established existence and uniqueness of the solution of the renewal equation arising in multistage game. We have proved it in a different method using contraction principle through a dynamic programming approach. . In the present model we have considered a dynamic model of renewal equation.

Key words: Multistage game, Dynamic programming, renewal equation, contraction principle, dynamic model.

INTRODUCTION:

Dynamic programming is a mathematical technique dealing with the optimization of multistage decision process which is based on "Bellmann's principle of optimality". As a result of this process some functional equation arise as a certain type of relationship between stage transformation and return function involving state and decision variables. Bhakta and Mitra [8], Bellmann have established a number of theorems for the existence and uniqueness of the solution of the functional equation arising in dynamic programming. I. Dmitry et al [4] studied a multi structural framework with dynamic considerations. We have established the existence and uniqueness of the solutions for a utility function. Bhardwaj et al. [6] summarized contractive mapping of different types and discussed on their fixed-point theorems. He considered many types of mappings and analyzed the relationship amongst them. In the class of multistage decision process where some decisions are made to maximize and some to minimize the objective function. This type of situation occurs in the theory of games. In the present work offers a two-person zero sum game, where loss of one player is equivalent to gain of the other.

Its mathematical analysis is fundamentally based upon the Min Max (Max Min) criteria of Von Neumann.

Let A and B be two protagonists. Suppose the first player A has a choice of M different plays and B has a choice of N different plays. We denote

a_{ij} = pay off to A if A chooses i^{th} and B chooses j^{th} alternatives.

b_{ij} = pay off to B if A chooses i^{th} and B chooses j^{th} alternatives.

Suppose A makes i^{th} choice with probability $p_i$and for B the j^{th} choice with probability q_j . The vectors $p = (p_1, p_2, \cdots, p_M)$, $q = (q_1, q_2, \cdots, q_N)$, specify the probability distribution of A and B respectively.

If B is required to choose q before A chooses p then the expected return to A is

$$V_A = \min_q \max_p \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij}\ p_i\ q_j$$

If the situation is reversed then expected return to B becomes

$$V_B = \max_p \min_q \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij}\ p_i\ q_j$$

The basic result in the theory of games derived from Min Max (Max Min) theorem of Von Neumann is that

$$V_A = V_B.$$

Its common value is the value of the game.

Suppose A and B choose from a continuous domain [0,1]. A chooses x and B chooses y from [0,1]. Let

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the payoff function k(x,y) measures the value to A and -k(x,y), the value to B. Let F(x) and G(y) be the distribution function to A and B respectively. Then the expected gain to A will be

$$V_A = \int_0^1\int_0^1 k(x,y)\, dF(x)\, dG(y)$$

Where

$$d(F) \geq 0\,, \quad \int_0^1 dF(x) = 1$$

$$d(G) \geq 0\,, \quad \int_0^1 dG(y) = 1.$$

Formulation of the Functional Equation:

Let us consider a two person zero sum multistage game. Suppose at any stage the vectors x and y are the state of the players A and B. At the beginning of each stage of an N-stage process A allocates a certain quantity u of his resources x and B allocates a certain v of his resorces y, where $0 \leq u \leq x,\ 0 \leq v \leq y$.

A receives a pay off $R(x,y;u,v)$ and B receives a pay off $-R(x,y;u,v)$. R is the set of real numbers. x is transformed into $T(x,y;u,v)$ and y into $T'(x,y;u,v)$, where $T, T' : S \times D \to R$. Let G and G' are the distribution functions defined over the region $0 \leq u \leq x,\ \ 0 \leq v \leq y$.

In the case of more general situation, let us now change our notation x and y to p and p' respectively. Then the corresponding recurrence relation is given by

$$f(p,p') = \max_G \min_{G'} \left[\int\int R(u,v)dG(u)dG'(v)\right],\quad 0 \leq u \leq p,\ 0 \leq v \leq p'$$

$$= \min_{G'} \max_G [\dots\dots\dots\dots]$$

$$f_{N+1}(p,p') = \max_G \min_{G'} \left[\int\int [R(u,v) + f_N(T,T')]dG(u)dG'(v)\right],\quad 0 \leq u \leq p,\ 0 \leq v \leq p'.$$

$$= \min_{G'} \max_G [\dots\dots\dots\dots]$$

Where $R(u,v) = R(u,v;p,p')$

Let us extend this idea to an infinite process. Then

$$f(p,p') = \max_G \min_{G'} \left[\int_u\int_v [\,R(u,v) + h(p,p';u,v)f(T,T')]\, dG(u)dG'(v)\right]$$

$$= \min_{G'} \max_G [\dots\dots\dots\dots] \qquad \dots\dots..(1)$$

Here we want to establish some existence theorems using contraction principle through a dynamic programming approach.

Lemma 1: If for $i = 1,2$

$$\Psi_i(x,y) = \max_G \min_{G'} \left[\int_u\int_v [\,R(u,v) + h(p,p';u,v)f(T,T')]\, dG(u)dG'(v)\right] = \min_{G'} \max_G [\dots\dots\dots\dots],$$

$$|\Psi_1(x,y) - \Psi_2(x,y)| \leq \max_u \max_v\ [|h(p,p';u,v)||f_1(T,T') - f_2(T,T')|\,]$$

$$\Psi_1(x,y) - \Psi_2(x,y) \leq \max_G \max_{G'} \int_u\int_v h(p,p';u,v)[f_1(T,T') - f_2(T,T')]dG(u)dG'(v)$$

$$|\Psi_1(x,y) - \Psi_2(x,y)| \leq \max_u \max_v [|h(p,p';u,v)||f_1(T,T') - f_2(T,T')|]$$

Lemma 2: Let(S,d) be a complete metric space . A is a mapping from S into itself. If the following conditions hold, then A has a unique fixed point.

i) For any $x, y \in S, d(Ax,Ay) \leq \emptyset(d(x,y))$, where $\emptyset: [0,\infty) \to [0,\ \infty)$ is non decreasing continuous on the right and $\emptyset(r) < r$ for $r > 0$.

ii) For every $x \in S$, there is a positive number λ_x such that $d(x, A^n x) \leq\ \lambda_x$ for all n.

Lemma 3: Let (S,d) be a complete metric space and let A be a mapping of S into itself satisfying $d(Ax,Ay) \leq \emptyset(d(x,y))$, for all x, y in S.

Where $\emptyset : [0,\infty) \to [0,\infty)$ is non decreasing and for every positive r, the series $\sum \emptyset^n(r)$ is convergent .Then A has a unique fixed point.

Existence theorems

Theorem 1: The functional equation (1) possesses unique bounded solution on S under the following conditions.

i) R and h are bounded.

ii) $\max_u \max_v [|h(p,p';u,v)||f_1(T,T') - f_2(T,T')|] \leq \lambda |f_1(T,T') - f_2(T,T')|$

Where $\lambda < 1$.

Proof: Let $B(S)$ be the set of all real valued bounded functions on S. For $\Psi_1, \Psi_2 \in B(S)$, $Let\ d(\Psi_1,\Psi_2) = \max_u \max_v [|\Psi_1(x,y) - \Psi_2(x,y)|]$. Then d is a metric on $B(S)$ and $(B(S), d)$ is a complete metric space. Let A be any function defined on $B(S)$ by $Ag = \Psi$ for any g $\in B(S)$,

$$\Psi(p,p') = \max_G \min_{G'} [\int_u \int_v [R(u,v) + h(p,p';u,v)g(T,T')] dG(u)dG'(v)]$$

$$= \min_{G'} \max_G [\dots\dots\dots]$$

Since R, h and g are bounded, Ψ is also bounded, and $\Psi \in B(S)$. Hence A maps $B(S)$ into itself.

Thus A maps B(S) into itself. Also any fixed point of A is a solution of the functional equation (1) and conversely any bounded solution of equation (1) is a fixed point of A.

Let $g_1, g_2 \in B(S)$ and $Ag_1 = \Psi_1, Ag_2 = \Psi_2$. Then for i=1,2

$$\Psi_i(p,p') = \max_G \min_{G'} [\int_u \int_v [R(u,v) + h(p,p';u,v)g_i(T,T')] dG(u)dG'(v)]$$

$$= \min_{G'} \max_G [\dots\dots\dots]$$

Let G_i, G_i' be the distribution functions yielding $\Psi_i(x,y)$ for i =1,2, then using lemma 1 , we have $|\Psi_1(x,y) - \Psi_2(x,y)| \le \max_u \max_v [|h(p,p';u,v)||g_1(T,T') - g_2(T,T')|] \le \lambda|g_1(T,T') - g_2(T,T')|$

i.e., $d(\Psi_1,\Psi_2) \le \lambda\, d(g_1,g_2)$ ………… (2)

Thus A is a contraction mapping. We define a mapping $\emptyset : [0,\infty) \to [0,\infty)$ by $\emptyset(r) = \lambda r\ \ for\ r > 0.\ Since\ \lambda < 1,\ \emptyset(r) < r$. $\emptyset$ is non decreasing on [0,∞).

Then equation (2) becomes $d(Ag_1, Ag_2) \le \emptyset\ d(g_1,g_2)$. Which is the condition of Lemma 2.

Again for g $\in B(S)$, set $g_n = A^n$ g, for $n = 1,2 \dots\dots$

$$g_n(p,p') = \max_G \min_{G'} [\int_u \int_v [R(u,v) + h(p,p';u,v)g_{n-1}(T,T')] dG(u)dG'(v)]$$

$$= \min_{G'} \max_G [\dots\dots\dots] \quad for\ n = 2,3 \dots\dots$$

$$g_1(p,p') = \max_G \min_{G'} [\int \int R(u,v)dG(u)dG'(v)]$$

$$= \min_{G'} \max_G [\dots\dots\dots]$$

Since R, h and g are all bounded functions, $|R(u,v)| \le \lambda_1$, $|h(p,p';u,v)g_n(T,T')| \le \lambda_2$ and $|g(p,p')| \le \lambda_3$. $\lambda_1, \lambda_2, \lambda_3$ are constants for all $(p,p') \in S, (u,v) \in D, g(T,T') \in R$

$Thus\ |g_n(p,p')| \le \lambda_1 + \lambda_2\ for\ all\ (p,p') \in S.$ $|g(p,p') - g_n(p,p')| \le \lambda_1 + \lambda_2 + \lambda_3 = \lambda, for\ all\ n.$

Thus $d(g, A^n g) \le \lambda$ for all n, which is the second condition of lemma 2.Thus the mapping A has a unique fixed point, i.e., the functional equation (1) has a unique bounded solution on S.

Theorem 2: Suppose that

$$\max_u \max_v [|h(p,p';u,v)||f_1(T,T') - f_2(T,T')|] \le \emptyset|f_1(T,T') - f_2(T,T')|$$

Where $\lambda < 1$ and $\emptyset: [0,\infty) \to [0,\infty)$ is non decreasing and continuous on the right such that $\emptyset(r) < r$ for $r > 0$. Then the functional equation (1) possesses a unique solution on S.

Proof: As in theorem 1, we define a function $\emptyset: [0,\infty) \to [0,\infty)$ by $\emptyset(r) = r\lambda$ for any $r > 0$. Then $\emptyset$ is non decreasing and continuous on the right. Again $\emptyset(r) < r$, since $\lambda < 1$.

By Lemma 1,

$$|\Psi_1(p,p') - \Psi_2(p,p')| \le \max_u \max_v |h(p,p';u,v)||g_1(T,T') - g_2(T,T')| \le \emptyset|g_1(T,T') - g_2(T,T')| = \lambda|g_1(T,T') - g_2(T,T')|$$

$\therefore d(\Psi_1,\Psi_2) \le \lambda\, d(g_1,g_2) = \emptyset(d(g_1,g_2))$

By Lemma 3, A has a fixed point and hence the theorem.

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