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RP:-171: Solving Standard Cubic Congruence modulo aProduct ofSpecial Even multiple of an Odd Prime and Powered Three.

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ABSTRACT
In this research paper, the author hasformulated the solutions ofa standard cubic congruence of composite modulus modulo a product of even multiple of an odd prime and powered three in three different cases. It is found that the said congruence has three types of solutions as per the case. In the first case, it has only 3 incongruent solutions; in the second case, it has exactly twelve incongruent solutions while in the third case, it has exactly nine incongruent solutions; p being an odd prime positive integer. Such types of congruence were not studied by the earlier mathematicians. Formulation of solutions has provided a simple procedure of finding the required solutions very easily. This is the merit of the paper.

KEY-WORDS: Cubic Congruence, Composite Modulus, Chinese Remainder Theorem, Formulation, Incongruent Solutions.

I. INTRODUCTION

The author is very fond of Number theory. He has considered many standard quadratic, cubic and bi-quadratic congruence for his study. The author has formulated the solutions of many such congruence successfully and published in many International Journals [1], [2], [3], [4],[5],[6].

PROBLEM-STATEMENT

To formulate the solutions of the standard cubic congruence:

$x^3 \equiv a^3 (\text{mod } 8p.3^n)$ with p an odd prime, n any positive integer.

II. LITERATURE-REVIEW

No study material is found in the literature of mathematics. Standard cubic congruence was not remain a part of the university syllabus. But more literature of quadratic congruence are found in the books of Number Theory [7], [8], [9]. It is the author's opinion that the readers may use the Chinese Remainder Theorem (C R T)[7]. But readers do not prefer it to use as it is very complicated. So the formulation is needed.

III. ANALYSIS & RESULTS

Consider the congruence: $x^3 \equiv a^3 (\text{mod } 8p.3^n)$.

Case-I: Let a be an odd positive integer.

For its solutions, consider $x \equiv 8p.3^{n-1}k + a \ (\text{mod } 8p.3^n)$

Then, $x^3 \equiv (8p.3^{n-1}k + a)^3 \ (\text{mod } 8p.3^n)$

$$\equiv (8p.3^{n-1}k)^3 + 3.(8p.3^{n-1}k)^2.a + 3.8p.3^{n-1}k.a^2 + a^3 \ (\text{mod } 8p.3^n)$$

$$\equiv 8p.3^n k\{8^2p^2.3^{2n-3}k^2 + 8p.3^{n-1}ka + a^2\} + a^3 \ (\text{mod } 8p.3^n)$$

$$\equiv 0 + a^3 (\text{mod } 8p.3^n), \text{as a is odd.}$$

$$\equiv a^3 (\text{mod } 8p.3^n).$$

Therefore, $x \equiv 8p.3^{n-1}.k + a \ (\text{mod } 8p.3^n)$ gives all the solutions of the said congruence.

But for $k = 3, x \equiv 8p.3^{n-1}.3 + a \ (\text{mod } 8p.3^n)$

$$\equiv 8p.3^n + a \ (\text{mod } 8p.3^n)$$

$\equiv 0 + a \ (\text{mod } 8p.3^n)$.This is the same solutionas for $k = 0$.

Also for $k = 4 = 3 + 1$, $x \equiv 8p.3^{n-1}.(3 + 1) + a \ (\text{mod } 8p.3^n)$

$\equiv 8p.3^n + 8p.3^{n-1} + a \ (\text{mod } 8p.3^n)$

$\equiv 8p.3^{n-1} + a \ (\text{mod } 8p.3^n)$.This is the same solutionas for $k = 1$.

Hence all the solutions are given by

$$x \equiv 8p.3^{n-1}.k + a \ (\text{mod } 8p.3^n); k = 0, 1, 2.$$

This gives exactly three incongruent solutions.

Case-II: Let a be an even positive integer.

For its solutions, consider $x \equiv 2p.3^{n-1}k + a \ (\text{mod } 8p.3^n)$

Then, $x^3 \equiv (2p.3^{n-1}k + a)^3 \ (\text{mod } 8p.3^n)$

$$\equiv (2p.3^{n-1}k)^3 + 3.(2p.3^{n-1}k)^2.a + 3.2p.3^{n-1}k.a^2 + a^3 \ (\text{mod } 8p.3^n)$$

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$\equiv 2p.3^n k\{2^2p^2.3^{2n-3}k^2 + 2p.3^{n-1} ka + a^2\} + a^3 \ (mod\ 8p.3^n)$

$\equiv 2p.3^n k\{4t\} + a^3 (mod\ 8p.3^n)$, as a is even.

$\equiv 0 + a^3 (mod\ 8p.3^n)$.

Therefore, $x \equiv 2p.3^{n-1}.k + a \ (mod\ 8p.3^n)$ gives all the solutions of the said congruence.

But for $k = 12 = 4.3, x \equiv 2p.3^{n-1}.4.3 + a \ (mod\ 8p.3^n)$

$\equiv 8p.3^n + a \ (mod\ 8p.3^n)$

$\equiv 0 + a \ (mod\ 8p.3^n)$.This is the same solutionas for $k = 0$.

Also for $k = 13 = 12 + 1, \ x \equiv 2p.3^{n-1}.(4.3 + 1) + a \ (mod\ 8p.3^n)$

$\equiv 8p.3^n + 2p.3^{n-1} + a \ (mod\ 8p.3^n)$

$\equiv 2p.3^{n-1} + a \ (mod\ 8p.3^n)$. This is the same solutionas for $k = 1$.

Hence all the solutions are given by

$x \equiv 2p.3^{n-1}.k + a \ (mod\ 8p.3^n); k = 0, 1, 2, 3, \ldots\ldots\ldots, 11.$

This gives exactly twelve incongruent solutions.

Case-III: Let $a = 3$.

Then the said congruence reduces to the form: $x^3 \equiv 3^3 (mod\ 8p.3^n)$.

For its solutions, consider $x \equiv 8p.3^{n-2}k + 3 \ (mod\ 8p.3^n)$

Then, $x^3 \equiv (\ 8p.3^{n-2}k + 3\ )^3 \ (mod\ 8p.3^n)$

$\equiv (\ 8p.3^{n-2}k\ )^3 + 3.(\ 8p.3^{n-2}k\ )^2.3 + 3.8p.3^{n-2}k.3^2 + 3^3 \ (mod\ 8p.3^n)$

$\equiv 8p.3^n k\ \{8^2p^2.3^{2n-6}k^2 + 8p.3^{n-2}\ k + \ 3\} + 3^3 \ (mod\ 8p.3^n)$...............(A)

$\equiv 8p.3^n k\{t\} + a^3 (mod\ 8p.3^n), if\ n \geq 3.$

$\equiv 0 + a^3 (mod\ 8p.3^n)$.

Therefore, $x \equiv 8p.3^{n-2}.k + a \ (mod\ 8p.3^n)$ gives all the solutions of the said congruence.

But for $k = 9 = 3^2, x \equiv 8p.3^{n-2}.3^2 + a \ (mod\ 8p.3^n)$

$\equiv 8p.3^n + a \ (mod\ 8p.3^n)$

$\equiv 0 + a \ (mod\ 8p.3^n)$.This is the same solutionas for $k = 0$.

Also for $k = 10 = 9 + 1, \ x \equiv 8p.3^{n-2}.(3^2 + 1) + a \ (mod\ 8p.3^n)$

$\equiv 8p.3^n + 8p.3^{n-2} + a \ (mod\ 8p.3^n)$

$\equiv 8p.3^{n-2} + a \ (mod\ 8p.3^n)$.This is the same solutionas for $k = 1$.

Hence all the solutions are given by $x \equiv 8p.3^{n-2}.k + a \ (mod\ 8p.3^n)$; $k = 0, 1, 2, 3, \ldots\ldots\ldots, 8$.This gives exactly nine incongruent solutions.

Case-IV: Let $n \leq 3$. Then from case-III: Equation (A), it is clear that $x \equiv 8p.3^{n-1}k + 3 \ (mod\ 8p.3^n)$ gives the solutions. It gives exactly three incongruent solutions for $k = 0, 1, 2$.

IV. ILLUSTRATIONS

Example-1: Consider the congruence: $x^3 \equiv 1 \ (mod\ 360\ )$.

It can be written as: $x^3 \equiv 1^3 (mod\ 8.5.9) \ \ i.e. x^3 \equiv 1^3 (mod\ 8.5.3^2)$.

It is of the type: $x^3 \equiv a^3 (mod\ 8p.3^n) \ \ with\ a = 1, p = 5, n = 2.$

It has exactly three incongruent solutions given by

$x \equiv 8p.3^{n-1}k + a \ (mod\ 8p.3^n)$

$\equiv 8.5.3^{2-1}k + 1 \ (mod\ (8.5.3^2)$

$\equiv 120k + 1 \ (mod\ 360); k = 0, 1, 2.$

$\equiv 0 + 1, 120 + 1, 240 + 1 \ (mod\ 360)$

≡1, 121, 241 (mod 360).

Example-2: Consider the congruence: $x^3 \equiv 8 \ (mod\ 360\ )$.

It can be written as: $x^3 \equiv 2^3 (mod\ 8.5.9) \ \ i.e. x^3 \equiv 2^3 (mod\ 8.5.3^2)$.

It is of the type: $x^3 \equiv a^3 (mod\ 8p.3^n) \ \ with\ a = 2, p = 5, n = 2.$

It has exactly twelve incongruent solutions given by

$x \equiv 2p.3^{n-1}k + a \ (mod\ 8p.3^n)$

$\equiv 2.5.3^{2-1}k + 2 \ (mod\ (8.5.3^2)$

$\equiv 30k + 2 \ (mod\ 360); k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.$

$\equiv 0 + 2, 30 + 2, 60 + 2, 90 + 2, 120 + 2, 150 + 2, 180 + 2, 210 + 2, 240 + 2, 270 + 2, 300 + 2, 330 + 2 \ (mod\ 360)$

≡2, 32, 62, 92, 122, 152,182, 212, 242, 272, 302, 332 (mod 360).

Example-3: Consider the congruence: $x^3 \equiv 125 \ (mod\ 756\ )$.

It can be written as: $x^3 \equiv 5^3 (mod\ 4.7.27) \ \ i.e. x^3 \equiv 5^3 (mod\ 4.7.3^3)$.

It is of the type: $x^3 \equiv a^3 (mod\ 4.p.3^n) \ \ with\ a = 5, p = 7, n = 3.$

It has exactly three incongruent solutions given by

$x \equiv 4p.3^{n-1}k + a \ (mod\ 4p.3^n)$

$\equiv 4.7.3^{3-1}k + 5 \ (mod\ (4.7.3^3)$

$\equiv 252k + 5 \ (mod\ 756); k = 0, 1, 2.$

$\equiv 0 + 5, 252 + 5, 504 + 5 \ (mod\ 756)$

≡5, 257, 509(mod 756).

Example-4: Consider the congruence: $x^3 \equiv 64 \ (mod\ 1512\ )$.

It can be written as: $x^3 \equiv 4^3 (mod\ 8.7.27) \ \ i.e. x^3 \equiv 4^3 (mod\ 8.7.3^3)$.

It is of the type: $x^3 \equiv a^3 (mod\ 8p.3^n) \ \ with\ a = 4, p = 7, n = 3.$

It has exactly twelve incongruent solutions given by

$x \equiv 2p.3^{n-1}k + a \ (mod\ 8p.3^n)$

$\equiv 2.7.3^{3-1}k + 4 \ (mod\ (8.7.3^3)$

$\equiv 126k + 4 \ (mod\ 1512); k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.$

$\equiv 0+4,126+4,252+4,378+4,504+4,630+4, 756+4,882+4,1008+4,1134+4,1260+4,1386+4 \ (mod\ 1512)$

≡ 2, 130,256,382, 508, 634, 760, 886, 1012, 1138, 1264, 1390 (mod 1512).

Example-5: Consider the congruence: $x^3 \equiv 27\ (mod\ 3240\ )$.

It can be written as: $x^3 \equiv 3^3(mod\ 8.5.81)\ \ i.e. x^3 \equiv 3^3(mod\ 8.5.3^4)$.

It is of the type: $x^3 \equiv 3^3(mod\ 8p.3^n)\ \ with\ a = 3, p = 5, n = 4.$

It has exactly nine incongruent solutions given by $x \equiv 8p.3^{n-2}k + a\ (mod\ 8p.3^n); k = 0,1,2,3,4,5,6,7,8.$

$$\equiv 8.5.9k + 3\ (mod\ (8.5.3^4)$$

$$\equiv 360k + 3\ (mod\ 3240); k = 0,1,2 \dots\dots\dots.., 8..$$

$$\equiv 0+3,360+3,720+3,1080+3,1440+3,1800+3,2160+3,2520+3,2880+3\ \ (mod\ 3240)$$

≡ 3, 363, 723, 1083, 1443, 1803, 2163, 2523, 2883 (mod 3240)

Example-6: Consider the congruence: $x^3 \equiv 27\ (mod\ 360\ )$.

It can be written as: $x^3 \equiv 3^3(mod\ 8.5.9)\ \ i.e. x^3 \equiv 3^3(mod\ 8.5.3^2)$.

It is of the type: $x^3 \equiv 3^3(mod\ 8p.3^n)\ \ with\ a = 3, p = 5, n = 2.$

It has exactly three incongruent solutions given by

$$x \equiv 8p.3^{n-1}k + a\ (mod\ 8p.3^n)$$

$$\equiv 8.5.3^{2-1}k + 3\ (mod\ (8.5.3^2)$$

$$\equiv 120k + 3\ (mod\ 360); k = 0,1,2.$$

$$\equiv 0+3,120+3,240+3\ (mod\ 360)$$

≡ 3, 123, 243 (mod 360)

Example-7: Consider the congruence: $x^3 \equiv 27\ (mod\ 1080\ )$.

It can be written as: $x^3 \equiv 3^3(mod\ 8.5.27)\ \ i.e. x^3 \equiv 3^3(mod\ 8.5.3^3)$.

It is of the type: $x^3 \equiv 3^3(mod\ 8p.3^n)\ \ with\ a = 3, p = 5, n = 3.$

It has exactly nine incongruent solutions given by $x \equiv 8p.3^{n-2}k + a\ (mod\ 8p.3^n); k = 0,1,2,3,4,5,6,7,8.$

$$\equiv 8.5.3k + 3\ (mod\ (8.5.3^3)$$

$$\equiv 120k + 3\ (mod\ 1080); k = 0,1,2.$$

$$\equiv 0+3,120+3,240+3,360+3,480+3,600+3, 720+3,843,960+3\ (mod\ 1080)$$

≡ 3, 123, 243, 363, 483, 603, 723, 963 (mod 1080).

V. CONCLUSION

Therefore, it is concluded that the special standard cubic congruence of composite modulus modulo a product of special even multiple of an odd prime and powered three:

$x^3 \equiv a^3(mod\ 8p.3^n), p\ an\ odd\ prime$ has exactly three incongruent solutions

$x \equiv 8p.3^{n-1}k + a\ (\ 8p.3^n); k = 0,1,2,\ \ if\ a$ is an odd positive integer.

But $x^3 \equiv a^3(\mathrm{mod}\ 8p.3^n)$, p an odd prime has exactly twelve incongruent solutions

$x \equiv 2p.3^{n-1}k + a\ (\ 8p.3^n); k = 0,1,2,\dots\dots\dots..,11,$ if a is an even positive integer.

But the congruence: $x^3 \equiv 3^3(\mathrm{mod}\ 8p.3^n)$ for $n \geq 3$, has exactly nine incongruent solutions given by $x \equiv 8p.3^{n-2}k + a\ (\mathrm{mod}\ 8p.3^n); k = 0,1,2,3,4,5,6,7,8.$

But if $n \leq 2$, the congruence has exactly three incongruent solutions.

REFERENCE

[1]. Roy, B. M., Formulation of solutions of standard cubic congruence of a special even composite modulus in a special case, International Journal of Engineering Technology Research and Management (IJETRM), ISSN: 2456-9348, Vol-04, Issue-09, Sep-20.

[2]. Roy, B. M., Formulation of standard cubic congruence of composite modulus modulo a powered even prime multiplied by a powered three in two special case,International Journal for Research Trends and Innovations (IJRTI), ISSN: 2456-3315, Vol-05, Issue-09, Sep-20.

[3]. Roy, B. M., A review and reformulation of solutions of standard cubic congruence of composite modulus modulo an odd prime power integer, International Journal for scientific development and Research (IJSDR), ISSN: 2455-2631, Vol-05, Issue-12, Dec-20.

[4]. Roy, B. M., Solving some special standard cubic congruence modulo an odd prime multiplied by eight, International Journal of scientific Research and Engineering Development (IJSRED), ISSN: 2581-7175, Vol-04, Issue-01, Jan-21.

[5]. Roy, B. M., Solving some special standard cubic congruence of composite modulus modulo a multiple of an odd prime, International Journal of Trend in scientific Research and Development (IJTSRD), ISSN: 2456-6470, Vol-05, Issue-04, May-21.

[6]. Roy B. M., Solving four standard cubic congruence modulo an even multiple of square of an odd prime, International Journal

for scientific development and Research (IJSDR), ISSN: 2455-2631, Vol-06, Issue-06, Jun-21.
[7]. Zuckerman H. S., Niven I., 2008, An Introduction to the Theory of Numbers, Wiley India, Fifth Indian edition, ISBN: 978-81-265-1811-1.
[8]. Thomas Koshy, 2009, Elementary Number Theory with Applications, Academic Press, Second Edition, Indian print, New Dehli, India, ISBN: 978-81-312-1859-4.
[9]. David M Burton,2012,Elementary Number Theory, Mc Graw Hill education (Higher Education), Seventh Indian Edition, New Dehli, India, ISBN: 978-1-25-902576-1.