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Higher Order Compact Finite Difference Schemes for SolvingSecond-Order Linear Two-Point Boundary Value Problems.

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ABSTRACT: This paper discusses Higher Order Compact Finite Difference Schemes and its application to Second-Order Linear Two-Point Boundary Value Problems with Dirichlet boundary conditions. Compact Finite Difference Schemes of order 4 and 6 were derived for the general Second-Order Linear Two-Point Boundary Value Problems in Ordinary differential equations and the schemes were proven to be convergent. Numerical experiments were conducted on the Helmholtz equation (A special class of Second order ordinary differential differential) and several Second-Order ODEs with constant and variable coefficients. The compact Schemes derived were implemented using the Maple programming language, and the result obtained when compared with the Second-Order central difference scheme and Exact solutions showed that the CFDM is numerically accurate for all tested problems even at a step size of (h = 0:1) even when the central difference scheme failed to meet its Second-Order accuracy.

KEYWORDS: Compact Finite Difference Schemes,Dirichlet boundary conditions, Ordinary differential equations.

I. INTRODUCTION

In the world of Mathematics, exact (or analytic) solutions do not always exist for some Initial or Boundary Value Problems in differential equations. It has thus been the goal of Mathematicians to develop methods that can generate Numerical solutions which approximates the exact/analytic solution to a reasonably high level of accuracy. For the past fifty years [9] three methods 'Finite difference method, Integral equation method, and Finite Element Method', have dominated the world of numerical techniques/methods for solving problems involving equations that is differential. Differential equations are being used as essential tools in Mathematical modelling of some physical systems such as wave propagation, heat flow, radiation transfer, diffusion, fluid dynamics, electromagnetism, elastic vibrations, population dynamics etc., [3] play a large role in Mathematical Modelling, hence in solving these problems, differential schemes based on first and second order widely use numerical methods because of the relative simplicity of their implementation. Achieving a more accurate solution requires a higher order finite difference for approximations of derivatives as against the finite difference schemes that is less accurate. The number of stencils/points used increases thus making the equation systems to be more complex as these stencils increases.

The two most significant classes of techniques that are numerical used to solve differential equations that are partial are the element and difference methods. The finite difference method being preferred particularly for hyperbolic differential equation that is partial especially those that are quasi-linear and admit discontinuous solutions, has a few (major) drawbacks [6]. Derived difference schemes on irregular grids (or network) using Integral interpolation (or balance) methods were being the earliest work that studied the finite difference method on irregular grids (network) was proposed to simulate an electric network, and the schemes significantly reduced the geometrical error, and also provided an efficient and united approach in handling internal and natural boundary conditions

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and this made a considerable advancement in the invention of finite difference methods.

II. THE COMPACT FINITE DIFFERENCE METHOD (CFDM)

The compact finite different quasilinearization method, for the first time, it uses the compact FD Schemes in both space and time to improve this method's accuracy [5]. Studies a sub diffusion system that is fractional based on the fourth-order using a difference scheme, both compact and high-order. The compact finite difference scheme demonstrates a convergence that is high order [7]. The paper makes use of mixed derivatives together with linear second – order differential equations, Numerical test results showed numbers that do not contradict the order of accuracy with a high bonus (Reynolds), furthermore the study reflects the rate of convergence ranked at sixth-order, a boundary condition refer to as Dirichlet and an auditory boundary (Robin) ranked at fifth-order [8].

The study solves both one and two dimensional using multigrid algorithm in combination with a higher-order compact finite difference scheme in a homogeneous Heimholtz equation. Findings from the research generated an accurate eighth-order approximation for grids. A pictorial illustration was exhibited to describe the eighth-order compact difference scheme accuracy and efficiency [4]. The study shows a pay-off that is not smooth for a convergence when pricing in models for stochastic volatility using a compact higher-order difference scheme [1].

The paper proposed a technique (integration method), more precise for simulation numerically of paraboline equations. Findings revealed that both the exact equation and proposed scheme had the same results, also replica type of compact schemes, CFDS – PIM, SMM generated better computational efficiency and the accuracy computationally of CFDD – PIM was higher when compared to C-N scheme in one-dimensional example[2].

III. METHODOLOGY.

Derivation of the Schemes

Given the general form of a second order Linear two-point boundary value problem

$$-u^{11}(x)+b(x)+c(x)u^{1}(x)=f(x).\ x\varepsilon I=[s,t] \quad (1)$$

$$u(s)=u_s,u(t)=u_t \qquad (2)$$

Where $u^{i}(x)=\dfrac{d^2u(x)}{dx}, u^{ii}(x)=\dfrac{d^2u(x)}{dx^2}$, and $I=[s,t]$,. The coefficients $b(x)$ and $c(x)$ in the equation (1) are given and belong to the Hilbert space $H^2(I)$ and the function $f(x)$ also belong to $H^2(I)$. To begin we discretise the interval $I=[s,t]$ into N equally spaced grid points with nodes $x_0,x_1,x_2x_3,...,x_{N-1},x_N$ then

$$I=\{s=x_0\prec x_1\prec x_2\prec x_3\prec ..\prec x_{N-1}\prec x_N=t\} \text{and}$$

$$x_i=x_0+ih, i=0,1,2,3,...,N$$

Where h is the spatial size. We make use of the following notations $u(x+nh)=u_{i+n}$ at $x=x_i$ then by Taylor's series expansion, we have that

$$u_{i+1}=u_i+hu^{I}+\frac{h}{2!}u_i^{II}+\frac{h^3}{3!}u_i^{(3)}+\frac{h^4}{4!}u_i^{(4)}+\frac{h^5}{5!}u_i^{(5)}+\frac{h^6}{6!}u_i^{(6)}+\frac{h^7}{7!}u_i^{(7)}+\dots \qquad (3)$$

$$u_{i-1}=u_i-hu^{I}+\frac{h}{2!}u_i^{II}+\frac{h^3}{3!}u_i^{(3)}+\frac{h^4}{4!}u_i^{(4)}+\frac{h^5}{5!}u_i^{(5)}+\frac{h^6}{6!}u_i^{(6)}+\frac{h^7}{7!}u_i^{(7)}+\dots \qquad (4)$$

Then denoting the second order central difference approximation for first and second derivatives of the function $u(x)$ by $\delta_x u_i$ and $\delta_x^2 u_i$, where the central difference approximations $\delta_x u_i$ and $\delta_x^2 u_i$ are given as

$$\delta_x u_i=\frac{u_{i+1}-u_{i-1}}{2h};\ \delta_x^2 u_i=\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} \qquad (5)$$

Substituting the Taylor's series expansion for u_{i+1} and u_{i-1} in (3) into (5) we have that

$$\delta_x u_i = u_i^1 + \frac{h^2}{3!}u_i^3 + \frac{h^4}{5!}u_i^{(5)} + \frac{h^6}{7!}u_i^{(7)} + \frac{h^8}{9!}u_i^{(9)} + \frac{h^{10}}{11!}u_i^{(11)} + \qquad (6)$$

$$\delta_x^2 u_i = u_i^{11} + 2\left(\frac{h^2}{4!}u_i^{(4)} + \frac{h^4}{6!}u_i^{(6)} + \frac{h^6}{8!}u_i^{(8)} + \frac{h^8}{10!}u_i^{(10)} + \frac{h^{10}}{12!}u_i(12) + \ldots\qquad (7)$$

So that

$$u_i^1 = \delta_x u_i - \left(\frac{h^2}{3!}u_i^3 + \frac{h^4}{5!}u_i^{(5)} + \frac{h^6}{7!}u_i^{(7)} + \frac{h^8}{9!}u_i^{(9)} + \frac{h^{10}}{11!}u_i^{(11)}\ldots\right), \qquad (8)$$

$$u_i^{11} = \delta_x^2 u_i - 2\left(\frac{h^2}{4!}u_i^{(4)} + \frac{h^4}{6!}u_i^{(6)} + \frac{h^6}{8!}u_i^{(8)} + \frac{h^8}{10!}u_i^{(10)} + \frac{h^{10}}{12!}u_i^{(12)} + \ldots\right) \qquad (9)$$

Substituting (8) and (9) into the boundary value problem (1) then we have

$$-\delta_x^2 u_i + b_i \delta_x u_i + c_i u_i + Err1 = f_i \qquad (10)$$

Where the term $Err1$ is the truncation error and it's given as

$$\text{Err1} = 2\left(\frac{h^2}{4!}u_i^{(4)} + \frac{h^4}{6!}u_i^{(6)} + \frac{h^6}{8!}u_i^{(8)}\right) + b_i\left(\frac{h^2}{3!}u_i^{(3)} + \frac{h^4}{5!}u_i^{(5)} + \frac{h^6}{7!}u_i^{(7)}\right) + \ldots$$

$$Err1 = \frac{2h^2\left(-2b_i u_i^{(3)} + u_i^{(4)}\right)}{4!} + \frac{2h^4\left(-3b_i u_i^{(5)} + u_i^{(6)}\right)}{6!} + \frac{2h^4\left(-4b_i u_i^{(7)} + u_i^{(8)}\right)}{8!} + \ldots \qquad (11)$$

So that the error term $Err1$ can be written as $Err1 = E_4 + E_6 + E_8 + \ldots$

IV. THE CENTRAL DIFFERENCE SCHEME

If we replace the terms $\delta_x^2 u_i$ and $\delta_x u_i$ in the equation (10) by the terms in (5), and neglecting the error term $(Err1)$ we obtain the following second order accurate central difference scheme.

$$A_i u_{i+1} + B_i u_i + C_i u_{i-1} = F_i,$$
$$i = 1,2,3,\ldots, N-1 \qquad (12)$$

and the coefficients $A_i, B_i, C_i = F_i$ are given as

$$A_i = \frac{-1}{h^2} + \frac{b_i}{2h}; \quad B_i = \frac{2_i}{2h} + c_i;$$

$$C_i = \frac{-1}{h^2} + \frac{-b_i}{2h}; \; F_i = f_i \qquad (13)$$

V. THE FOURTH ORDER COMPACT SCHEME

Thus to yield fourth order accuracy we only need to add the term E_4 to the second order central difference scheme given in (12) i.e. from (11)

$$E_{4i} = \frac{h^2\left(-2b_i u_i^{(3)} + u_i^{(4)}\right)}{12} \qquad (14)$$

The terms $u_i^{(3)}$ and $u_i^{(4)}$ are obtained by rearranging the equation (1) and differentiating repeatedly. Thus from (1) we have

$$u'' = b_i u_i' + c_i u_i - f_i \qquad (15)$$

This leads to

$$\left(-1+\frac{h^2c_i}{12}+\frac{h^2b_i'}{6}-\frac{h^2b_i^2}{12}\right)\delta_x^2u_i$$
$$+\left(b_i+\frac{h^2c_i}{6}+\frac{h^2b_i''}{12}-\frac{h^2c_ib_i}{12}-\frac{h^2b_i'b_i}{12}\right)\delta_x^2u_i$$
$$+\left(c_i+\frac{h^2c_i^i}{12}-\frac{h^2c_i^ib_i'}{12}\right)u_i+\frac{h^2b_if_i'}{12}-\frac{h^2f_i''}{12}+Err3=f_i \qquad (16)$$

Replacing the terms δ_x and δ_i^2 in (16) with their definitions in (5) and neglecting the error term, we obtain the following Fourth order compact finite difference scheme given as

$$A_iu_{i+1}+B_iu_i+C_iu_{i-1}=F_i,$$
$$i=1,2,3,\ldots,N-1 \qquad (17)$$

VI.THE SIXTH ORDER COMPACT SCHEME

To obtain a sixth order accurate scheme, we write the new error term Err3 of the fourth order compact difference scheme in the form Err3 = $E_6 + E_8 + E_{10} + \ldots$ Then following the same procedure as before we add only the term $E_6$to the fourth order compact scheme, which will in turn produce a new error term to be used for compact difference schemes of higher order. Thus proceeding as before,

$$E_6=h^4\left(\frac{2\left(-13b_iu_i^{(5)}+u_i^{(6)}\right)}{6!}-\frac{k_1u_i^{(3)}}{3!}-\frac{2k_1u_i^{(4)}}{4!}\right)$$

Using the same methods as before then the sixth order CFDM is given as;

$$A_iu_{i+1}+B_iu_i+C_iu_{i-1}=F_i, \quad i=1,2,3,\ldots,N-1 \qquad (18)$$

VII. APPLICATIONS

Application to problems with constant coefficient

Problem 1:

The Helmholtz equation

The Helmholtz equations are in the description of physical phenomena, such as acoustic, electromagnetic waves, elastic, etc [10]. The one dimensional Helmholtz equation takes the form $u^{11}(x)+k^2u(x)=f(x)$ where k is a constant. Thus

with $k=\frac{1}{3}$ and $f(x)=x^2+e^x$, we have the following

$$u^{11}(x)+k^2u(x)= \quad x^2+e^x \quad , \quad x\in[0,1]$$

With homogeneous boundary condition

u(0) = 0 , u(1) = 0

Exact solution of problem 1

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$$u(x) = c_1 \sin\left(\frac{x}{3}\right) + c_2 \cos\left(\frac{x}{3}\right) + 9x^2 + \frac{9e^x}{10} - 162$$

$$c_1 = \frac{-9\left(179\cos\left(\frac{1}{3}\right) - 170 + e\right)}{10\sin(\sin 13)} \; .,\qquad c_2 = \frac{1611}{10}$$

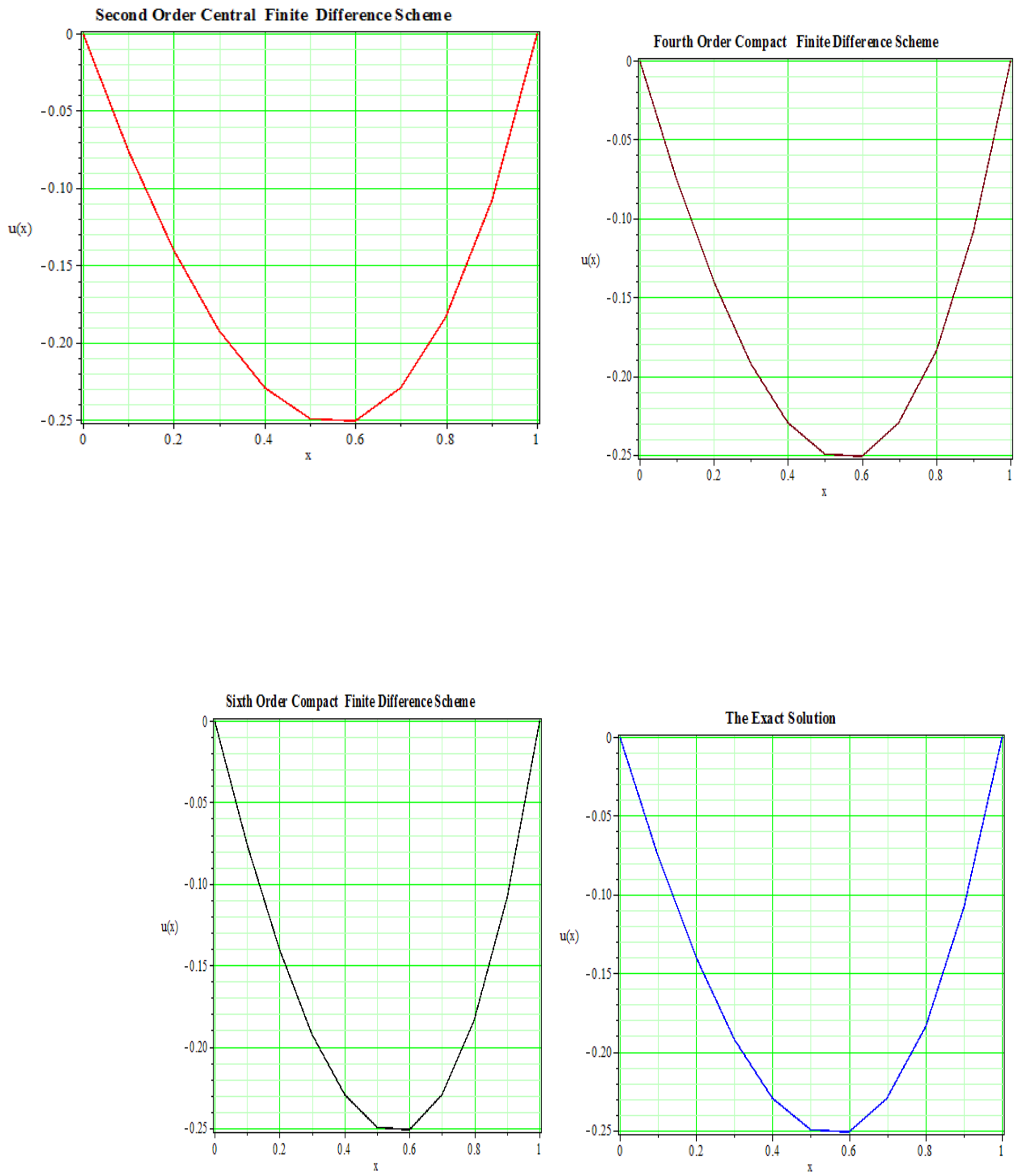

Figure 1: Plot of Distinct solutions for Problem 1

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Problem 2:

$$-u^{11}(x)+\frac{2}{x+1}u^{1}(x)+1-\frac{2}{(x+1)^2}u(x)=4x(1+x)e^{x}\ , \qquad x\in[0,1]$$

With homogeneous boundary condition
U(0) = 0 , u(0) = 0
Exact solution of problem 3

$$u(x)=x(1-x^2)e^{x}$$

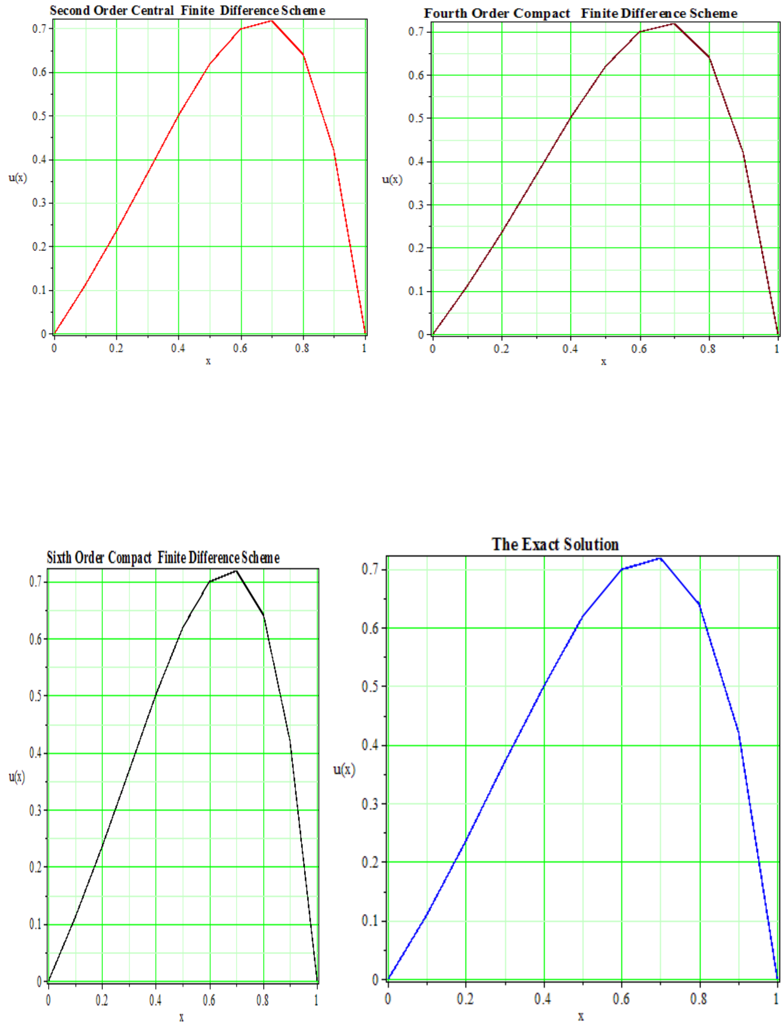

Figure 2: Plot of Distinct solutions for Problem 2

Problem 3:

$$(x+1)^2u^{11}(x)+(x+1)u^{1}(x)+u(x)=$$
$$4\sin(\ln(x+1));x\neq-1,x\in[0,4]$$

With boundary condition u(0) = 0 , u(4) = o

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Exact solution of problem 4 $u(x) = c_1 \sin(\ln(x+1)) + c_2 \cos(\ln(x+1)) + 2\sin(\ln(x+1)) - 2\ln(x+1)\cos\ln(x+1))$

$$c_1 = \frac{2(\ln(5))\cos(\ln(5)) - \sin(\ln(5))}{\sin(\ln(5))} \quad , \quad c_2 = 0$$

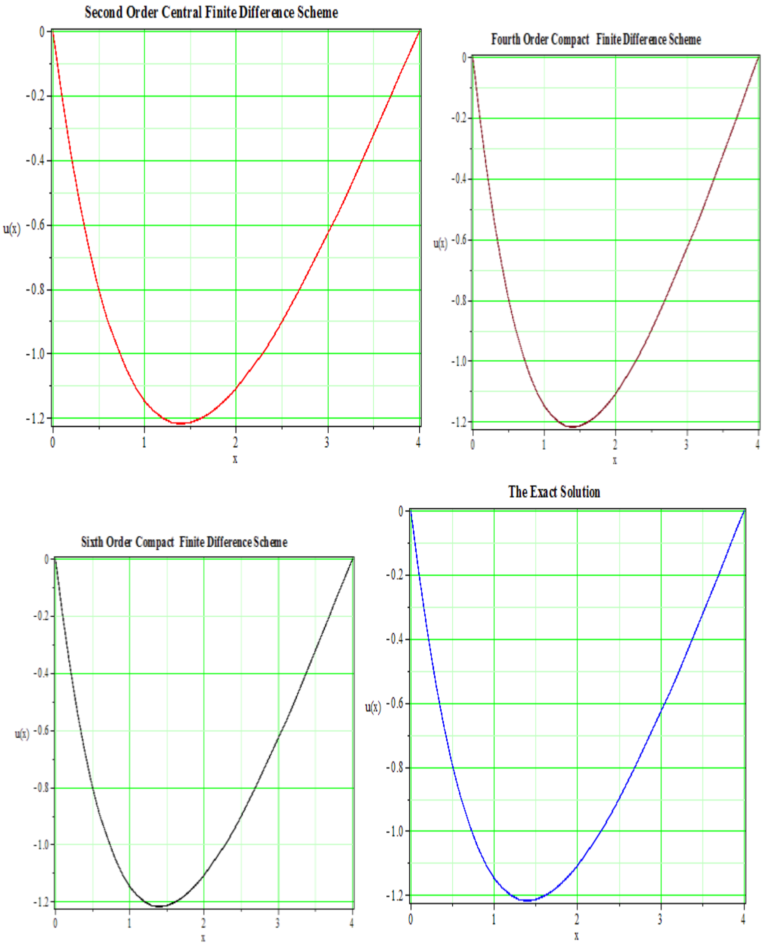

Figure 3: Plot of Distinct solutions for Problem 3

VIII.DISCUSSION AND CONCLUSION

The result obtained in problems I, 2 and 3 in comparison with the central difference scheme based on second-order compact finite difference scheme and Exact solutions showed that the CFDM is numerically accurate for fourth order compact finite difference scheme and sixth order compact finite difference scheme for all tested problems even at a step size of (h = 0:1) even when the central difference scheme failed to meet its second-Order accuracy. This can be showed from figure 1, 2 and 3 respectively.

In view of the obtained numerical results above and comparison with exact solutions, the following conclusions were deduced;

1. The schemes met the desired accuracy even at a mesh (step) size of h = 0.1 for all problems.
2. The accuracy order increases as the step size h decreases (i.e as $h \to 0$)

3. The error obtained at each nodal point is roughly constant in size thus indicating numerically stable schemes.

4. The experiments performed numerically whencompared with the exact solution clearlyreveals that the compact finite difference scheme greatlyexceeds the central difference scheme with a minimum accuracy of order 2 and 4 for thecompact scheme of order 4 and 6 respectively

5. Each new derivation from the Compact finite difference schemes exceeds the previous schemeby an accuracy of order 2, thus thecompact

finite difference schemes can beimproved to an arbitrarily large accuracy
order

REFERENCES

[1]. Bertram During and Michael Fournie (2012) higher-order compact finite difference scheme for option pricing in stochastic volatility models.Journal of computational and AppliedMathematics 236 (2012) 4462 – 4473.

[2]. Changkaic Chen, Xiaohua Zhang, Zhang Liu andYage Zhang (2020) A new high order Compact finite difference scheme based on precise integration method for the numericalSimulation of parabolic equations.Advance in Difference equation

[3]. Cheema.,T. A. (1997.) Higher-order finite difference methods for partial differentia equations.

[4]. FazalGhaffar, SaifUllah and Noor Badshah(2020) Multigrid method with eighth-order Compact finite difference scheme for Heimhotzequation. Physical Scripta ,Volume 95 Number 5.

[5]. P.GDlanmini and M Khunilo (2017): A new compact finite difference quasilinearization Method for non-linear evolution partialdifferential equation, Open Mathematics (15) IJBE/0118.

[6]. Ronghua Li, Z. C. & Wu., W. (2000.) Generalized difference methods for differential equations.

[7]. SeakengVung and Zhibo Wang (2015)Compact finite difference scheme for the fourth Order fractional sub-diffussion system: Advances inApplied Mathematics and Mechanics.

[8]. Spike Lee Jun Liu and Hai-Wai Sun (2014)combined compact difference scheme for linear Second-order partial differential equations withmixed derivative. Journal of Computational And Applied Mathematics 264 (2014) 23-37.

[9]. Tyler., J. G. (2007), Analysis and implementation of high-order compact finite differenceschemes. Master's thesis, Brigham Young University - Provo,.wikipedia (2017).

[10]. XiufangFeng, Z. L. &Qiao., Z. (2011), `High order compact finite difference schemesforthehelmholtz equation with discontinuouscoefficients.Journal of Computational Mathematics,Vol.29(No.3), 324340.