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Applications of Exponential Functions

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ABSTRACT: One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of the applications.

KEY-WORDS: Growth, decay, Radioactive decay, compound interest, Newton's law, Exponential growth and decay.

I. INTRODUCTION:

In addition to linear, quadratic, rational and radical functions, there are exponential functions. Exponential functions have the form $f(X) = b^x$, Where $b>0$ and $b\neq1$. Just as in any exponential expression, b is called the base and X is called the exponent.

An example of an exponential function is the growth of bacteria. Some bacteria double every hour. If you start with 1 bacterium and it doubles every hour , you will have 2^x bacteria after X hours. This can be written as $f(x) = 2^x$.

Before you start , $f(0)=2^0 =1$

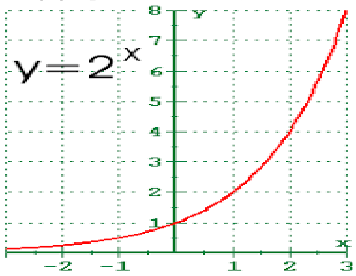
After 1 hour $f(1)=2^1=2$

In 2 hours $f(2)=2^2=4$

In 3 hour $f(3) = 2^3 =8$ so on

With the definition $f(x) =b^x$ and the restrictions that $b >0$ and that $b \neq1$, the domain of an exponential function is the set of all positive real numbers.

The following graph shows $f(x) = 2^x$.



EXPONENTIAL GROWTH AND DECAY

LEARNING OBJECTIVES

1. Use the exponential growth model in applications including population growth and compound interest.
2. Explain the concept of doubling time.
3. Use the exponential decay model in applications including radioactive decay and Newton's law of cooling.
4. Explain the concept of half – life.

EXPONENTIAL GROWTH MODEL

Many systems exhibit exponential growth. These systems follow a model of the form $y= be^{kt}$

Differentiating with respect to 't' we get

$\frac{dy}{dt} = k\, be^{kt}$

$y' = k\, be^{kt}$

$y' = k\, y$ $(y = be^{kt})$

That is the rate of growth is proportional to the current function value .This is a key feature of exponential growth , The equation $y' = k\, y$ involves derivatives and the equation called a differential equation .

EXPONENTIAL GROWTH:

Exponential growth is a specific way that a quantity may occurs increase over time . It occurs when the instantaneous rate of change of a quantity with respect to time is proportional to the itself .Describe as a function , a quantity undergoing exponential growth is an exponential function of time ,that is the variable representing time is the exponent .

EXPONENTIAL GROWTH FORMULAE:

Systems that exhibit exponential growth increase according to the mathematical model.

$Y(t) =y_0\, e^{kt}$

Where y_0 represents the initial state of the system and $k>0$ is called the growth constant. We can solve this equation by using separation of variables.

Population growth is a common example of exponential growth.

POPULATION GROWTH:

Population growth is the increasing growth of a population due to reproducing .A population growth rate is a rate at which a population increase every year or per time period that is being analyzed. Typically population growth is exponential however, at some point all population hit a tipping point where they can't support their growth rate any longer due to many factors including health and food supply .

POPULATION GROWTH MATH:

1. Change in population =Births –deaths
2. Per capita birth rate =b
3. Per capita death rate =d
4. # of individuals =N
5 . Rate of population growth (r) =b – d
6 .Survivor ship =% surviving.

POPULATION GROWTH FORMULAE

$P = P_0 e^{rt}$

P = Total population after time t

P_0 =Starting population

r =% Rate of growth

t =Time in hours or years

e = Euler number =2.71828…..

POPULATION GROWTH RATE:

Population growth is the percentage change in the size of the population in a year.

It is calculated by dividing the number of people added in a year per starting population, times 100

$$\text{Population growth rate} = \frac{\text{natural increase} + \text{net in migration}}{\text{Starting population}} \times 100$$

<u>POPULATION GROWTH CONCEPTS:</u>

<u>NET MIGRATION:</u> The difference between the numbers moving in and the numbers moving out of a defined area .

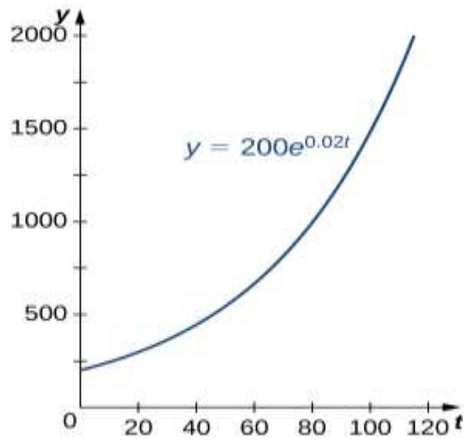
<u>NATURAL INCREASE:</u> The difference between the number of births and deaths in a difference population .

NOTE :

1 . When modeling a population with an exponential growth model , if the relative growth rate k is unknown , it should be determined . This is usually done using the known population at two particular times .

2 . Exponential growth models are good predictors for small populations in large populations with abundant resources , usually for relatively short time periods .

3 . The graph of the exponential growth $y(t) = y_0e^{kt}$ has the general form .



Where $y_0 = 200$, k= 0.02

The another example of an exponential growth is compound interest .

COMPOUND INTEREST:

Compound interest is the interest on a loan or deposit calculated based both the initial principal and the accumulated interest from previous periods . Thought to have originated in 17 th century at Italy . compound interest can be thought of as "interest on interest" and will make a sum grow at a faster rate than simple interest , which is calculated only on the principal amount .

CALCULATING COMPOUND INTEREST:

Compound interest is calculated by multiplying the initial principal amount by one plus the annual interest rate raised to the number of compound periods minus one . The total initial amount of the loan is than subtracted from the resulting value.

COMPOUND INTEREST FORMULA:

The formula for calculating compound interest is

Compound interest = Total amount of principal and interest in future less principal amount at present .

That is $C.I = p\,[\,(1+i)^n - 1\,]$

Where p = principal

i =nominal annual interest rate in percentage terms

n = number of compounding periods .

EX : Take a 3 years of 10,000 Rs at an interest rate of 5% that compounds annually .What would be the amount of interest ?

Solution : p = 10,000 i = 5\100 , n=3

$C.I. = 10{,}000\{\,[1 + 0.05\,]^3 - 1\,\}$

$= 10{,}000\,[\,(1.05)^3 - 1\,]$

ADVANTAGES OF COMPOUND INTEREST:

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1.Compound interest makes a sum of money grow at a faster rate than simple interest because in addition to earning returns on the money you invest .
2 .Also earn returns on those returns at the end of every compounding period , which could be daily, monthly , quarterly or annually .

RELATION SHIP BETWEEN COMPOUND INTEREST AND EXPONENTIAL GROWTH:

In finance compound returns cause exponential growth . The power of compounding is one of the most powerful forces in finance . This concept allows investors to create large sums with little initial capital. Solving accounts that carry a compound interest rate are common examples of exponential growth .

DIFFERENCE BETWEEN COMPOUND INTEREST AND EXPONENTIAL GROWTH:

Understanding compound growth lets you evaluate your business growth from year to year as results fluctuate .If you take a longer view with big goals for your business .

Exponential growth results show the long term potential if you can manage to produce compound growth year after year .

EXPONENTIAL DECAY:

A quantity is subject to exponential decay . If it decreases at a rate proportional to its current value . Symbolically this process can be expressed by the following differential equation where N is the quantity and λ is a positive rate called the exponential decay constant or rate constant .

FORMULA OF EXPONENTIAL DECAY:

The equation that describes exponential decay is $\frac{dN}{dt} = -\lambda N$

By using separation of variable we get $\frac{dN}{N} = -\lambda dt$

Integrating we have $\int \frac{dN}{N} dt = \int -\lambda dt$

$log\, N = -\lambda t + c$

W here c is the constant of integration . Hence we get $N(t) = e^{-\lambda t} . e^{c} = N_0 . e^{-\lambda t}$

Where $N_0 = e^{c}$

The final solution is $N(t) = e^{-\lambda t}$

This is the form of the equation i.e. most commonly used to describe exponential decay .In the above formula N (t) is quantity at time t , $N_0 = N(0)$ is the initial quantity , that is quantity at t = 0 and λ is Called decay constant (or) rate constant .

EXAMPLES OF EXPONENTIAL DECAY:

Examples of exponential decay are radioactive decay and population decrease . The information .for a city or colony in the future .

DEFINATION, HALF - LIFE:

If a quantity decays exponentially, the half life is the amount of time . It takes the quantity to be reduced by half . It is given by HALF – LIFE = $\frac{l\,n^2}{k}$

USES OF HALF – LIFE:

1. Half – life is the time required for a quantity to reduce to half of its initial value.
2 .The term is commonly used in nuclear physics to describe how quickly unstable atoms undergo or how long stable atoms survive , radioactive decay.
3. The term is also used more generally to characterize any type of exponential or non – exponential decay.

For ex: The medical sciences refer to the biological half – life of drugs and other chemicals in human body.

HALF –LIFE FORMULA:

One can describe exponential decay by any of the three formulas

$N(t) = N_0 ({}^1/_2) t/t_{1/2}$

$N(t) = N_0\, e^{-t}/\tau$

$N(t) = N_0\, e^{-\lambda t}$

Here N_0 refers to the initial quantity of the substance that will decay . T he measurement of this quantity may take place in grams ,moles , number of atoms etc .

N (t) is the quantity that still remains and its decay has not taken place after a time t . $t^{1/2}$ represents the half life of the decaying quantity is a positive number and is the mean life time of the decaying quantity .

λ is positive number and is creating the decay constant of the decaying quantity . The direct relation between the three parameters $t^{1/2}$, and λ is t_(1/2) = ln2/λ = ln2

Where ln 2 happens to be the natural logarithm of 2 (approx 0.693)

HALF - LIFE FORMULA DERIVATION:

We start from the exponential decay law which as follows $N(t) = N e^{-\lambda t}$ -------------------1

Let t = $T_{1/2}$ and $N(T_{1/2}) = \frac{1}{2} N_0$ ------------- --------------------------- 2

From 1 and 2 we get $N\ (T_{1/2}) = N_0\ e^{-\lambda t}$ --3

Sub 2 in 3 we get $\frac{N_0}{2} = N_0\ e^{-\lambda T_{1/2}}$

$\frac{1}{2} = e^{-\lambda T_{1/2}}$

Take logarithm we get $\ln 1/2 = \ln e\, e^{-\lambda T_{1/2}}$

$\ln 1/2 = -T_{1/2}$

$T_{1/2} = -\frac{1}{\lambda} \ln 1/2 = -\frac{\ln 2^{-1}}{\lambda} = -(-)\frac{\ln 2}{\lambda} = \frac{\ln 2}{\lambda}$

$T_{1/2} = \frac{\ln 2}{\lambda}$

The decay constant is given by $T_{1/2} = \frac{0.693}{\lambda}$

RADIO ACTIVE DECAY:

The rate of nuclear decay is also measured in terms of half lives . The half – life is the amount of time it takes for a given isotope to lose half of its radioactivity .

The Radioactivity decay law states that the probability per unit time that a nucleus will decay is a constant ,independent of time . This constant is called decay constant . It is denoted by λ . This constant probability may vary greatly between different types of nuclei ,leading the many different observed decay rates . The radioactive of certain no. of atoms (mass) is exponential in time .

FORMULA OF RADIOACTIVE DECAY:

Radioactive decay law $N = N_0\ e^{-\lambda t}$

Where N = the total no of particles in the sample .

λ = decay constant

t = time

TYPES OF RADIOACTIVE DECAY:

Namely 1 . Alpha
2 .Beta
3. Gamma

1 . ALPHA DECAY : When an alpha particle emit its nucleus ,the process is called alpha decay The formula of alpha decay is $E = (m_i - m_f - m_p)\ c^2$

Where m_i is the initial mass of the nucleus

m_f is the mass of the nucleus after particles emission

m_p is the mass of the emitted particle

2 . BETA DECAY : A beta particle is often referred to as an election ,but it can also be a position , if the reaction involves electrons ,nucleus shed out neutrons one by one . Even the proton number increases accordingly . A beta decay process is shown below .

Th^{234}_{90} ---------------------------------> $pa^{234}_{91} + e^{0}_{-1}$

GAMMA DECAY : The nucleus has orbiting electrons which indeed have some energy ,and when an electron jumps from a level of high energy to a level of low energy ,there is an emission of a photon . the same thing happens in the nucleus ,whenever it rearranges into a lower energy level ,a high –energy photon is shooted out which is known as a gamma decay .

LAW OF RADIOACTIVE DECAY DERIVATION :

The mathematical representation on of the law of radioactive decay is $\frac{\Delta N}{\Delta t} \propto N$

Where N is the total no of nuclei in the sample.

ΔN the number of nuclei that undergoes decay.

Δt is the unit time

$\frac{\Delta N}{\Delta t} = \lambda N$ where λ is the radioactive decay constant.

The change in the sample with respect to the numbers of nuclei is given as

$\frac{dN}{dt} = -\lambda N$

By using separation of variables

We get $\frac{dN}{dt} = -\lambda\, dt$

Integrating RHS over t_0 to t over and L H S over N_0 to N we get

$\int_{N_0}^{N} \frac{dN}{N} = \lambda \int_{t_0}^{t} dt$

$\log N - \log N_0 = -\lambda\ (t - t_0)$ Where N_0 is no . of radioactive nuclei and t_0 is arbitrary time .

$\log \frac{N}{N_0} = -\lambda t$

$N(t) = N_0\ e^{-\lambda t}$ is the law of radioactive decay .

The another example of exponential decay is Newton's law of cooling.

We are discussing def of Newton's formula, Derivation, Limitations. Examples

What is Newton's Law of Cooling?

Newton's law of cooling describes the rate at which an exposed body changes temperature through radiation which is approximately proportional to the difference between the object's temperature and its surroundings, provided the difference is small.

Definition: According to Newton's law of cooling the rate of loss of heat from a body is directly

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proportional to the difference in the temperature of the body and its surroundings.

Table of Content:

- Fomula
- Derivation
- Limitations
- Solved Examples

Newton's law of cooling is given by, dT/dt =k(T. - T.)

Where,

T_t = temperature at time t

T_s = temperature of the surrounding,

k= Positive constant that depends on the area and nature of the surface of the body under consideration.

Newton's Law of Cooling Formula

Greater the difference in temperature between the system and surrounding, more rapidly the heis transferred i.e. more rapidly the body temperature of body changes. Newton's law of cooling formula is expressed by,$T(t) = T_s + (T_0 – T_s) e^{-kt}$

Where

- t = time,
- T(t) = temperature of the given body at time t,
- T_s = surrounding temperature,
- T_0= initial temperature of the body, .
- k=constant

Newton's Law of Cooling Derivation

For small temperature difference between a body and its surrounding, the rate of cooling of body is directly proportional to the temperature difference and the surface area exposed.

$dQ/dt \propto (q – q_s)$, where q and q_s are temperature corresponding to object and surroundings

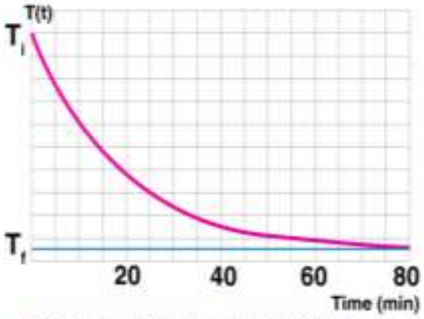
From above expression , $dQ/dt = -k[q – q_s]$ . . . . (1)

This expression represents Newton's law of cooling. It can be derived directly from Stefan's law, which gives,

$k = [4e\sigma\times\theta^3_o/mc] A$ . . . . . (2)

Now, $d\theta/dt = -k[\theta – \theta_o]$

⇒ \int_{\theta_1}^{\theta_2}\frac{d\theta}{(\theta-\theta_o)} = \int_{0}^{1}-k dt∫θ1θ2(θ−θo)dθ=∫01 −kdt



Newton's Law of Cooling – Temperature vs Time

where,

q_i = initial temperature of object,

q_f = final temperature of object.

$\ln (q_f – q_0)/(q_i – q_0) = kt$

$(q_f – q_0) = (q_i – q_0) e^{-kt}$

$q_f = q_0 + (q_i – q_0) e^{-kt}$ . . . . . . (3).

Methods to Apply Newton's Law of Cooling

Sometime when we need only approximate values from Newton's law, we can assume a constant rate of cooling, which is equal to the rate of cooling corresponding to the average temperature of the body during the interval.

i.e. $d\theta\backslash dt = k(\langle q\rangle – q_0)$ . . . . (4)

If q_i and q_f be the initial and final temperature of the body then,

$\langle q\rangle = (q_i + q_f)/2$ . . . . . (5)

Remember equation (5) is only an approximation and equation (1) must be used for exact values.

Limitations of Newtons Law of Cooling

- The difference in temperature between the body and surroundings must be small,
- The loss of heat from the body should be by radiation only,
- The major limitation of Newton's law of cooling is that the temperature of surroundings must remain constant during the cooling of the body.

II. CONCLUSION:

We have seen the application of exponential functions. we are using exponential functions growth and decay. It show up in a host of natural applications. From population growth and continuously Compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature.

BIBLIOGRAPHY:

[1]. Differential Equations and their applications second edition Author: Zafar Ahsan

[2]. O.D.E and P.D.E ,Author M.D Raisingania19" Edition

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[3]. Contributed by Gilbert strang and Edwin "jed Herman from Maths Libre Texts.
[4]. ALIYAHASAN (USMANU DANFODIYO UNIVERSITY), YAHAYA ZAKARI (AHMADU BELLO UNIVERSITY) PROJECT :"APPLICATIONS OF DIFFERENTIAL EQUATIONS

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