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A Survey of Some Polynomial Basis Functions for the Convergence of the Approximate Solution of Volterra Integral Equations of the First Kind

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ABSTRACT
This paper is aimed at verifying whether or not the choice of a polynomial basis function affects the convergence of the approximate solution to the exact solution for a Volterra integral equation of the first kind using the Galerkin's residual method. Five different polynomials basis functions were considered in this work. The procedure resulted in the construction of different systems of algebraic equations for each value ofn; the approximate solutions$\bar{y}(x)$ obtained coincide with the exact solutions at the same value of n as can be seen in problems 1-3, suggesting that the solution does not depends largely on the choice of the polynomial basis function except in the case of Laguerre polynomials where the approximate solution does not coincide with the exact solution for all values ofn in all the examples considered.

Keywords:-Chebyshev polynomials, Legendre polynomials, Laguerre Polynomials, Bernoulli Polynomials, Bernstein Polynomials.

I. INTRODUCTION

The applications of Volterra integral equations are found in the study of the risk of insolvency in actuarial science. Volterra integral equations arise in various fields such as Physics, Engineering and Economics and because of the role of these fields in the development of the society and all mankind in recent time's mathematicians and scientist alike are motivated to the development of reliable methods for solving this equation (see Reinkenhof, (1977), Kreyszig, (1979), Mandal and Bhattacharya (2007, 2008)). Approximation methods for solving various classes of Volterra integral equations (VIEs) are very rare (see Cicelia, (2014), Altürk (2016) andBellour and Rawashdeh (2010)). Several methods have been proposed for the numerical solutions of these equations.

Numerical solutions of VIEs have been studied by many scientist using different approaches and methods. The Numerical Solutions of VIE using Laguerre Polynomials was discussed by Rahman et al., (2012), Bernstein polynomials was used in the approximation techniques for VIE (see Altürk (2016), Wang and Wang (2014) and Kreyszig (1979)). Taylor series polynomials were used for the numerical solution of VIE by Wang and Wang (2014).

In this paper, the introduction of the Chebyshev polynomials and Galerkin's residual method which will helps in transforming the VIE to linear systems of algebraic equations whose matrices are easily handled shall be considered using the Scientific Workplace software.

II. METHODS

2.0 Bernoulli Polynomials:

The Bernoulli polynomials up to degree n can be defined over the interval [0,1]implicitly by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, x^{n-k},\ \ i = 0,1,\dots,n \qquad (1)$$

where B_k are the Bernoulli number given by

$B_0 = 1$ and $B_k = -\int_0^1 B_k dx\ ,\ k \geq 1$

These Bernoulli polynomials may be obtained explicitly as

$$B_0(x) = 1$$

$$B_m(x) = \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^m - \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k)^m\ ,\ \ m \geq 1 (2)$$

Below are some of the first few Bernoulli polynomials obtained from recurrence relation (2)

$$B_0(x) = 1, B_1(x) = x,$$
$$B_2(x) = -x + x^2,\ \ B_3(x) = \frac{x}{2} - \frac{3x^2}{2} + x^3$$

$B_4(x) = x^2 - 2x^3 + x^4$, $B_5(x) = -\frac{x}{6} + \frac{5x^3}{3} - \frac{5x^4}{2} + x^5$, $B_6(x) = x^6 - 3x^5 + \frac{5x^4}{2} - \frac{1}{2}x^2$

2.1 Laguerre Polynomials:

The general form of the Laguerre polynomials of n^{th} degree is defined by

$L_n(x) = \sum_{i=0}^{n}(-1)^i \frac{(n!)^2}{(n-i)!(i!)^2} x^i, i = 0.1, \dots, n$ (3)

Using Scientific Workplace package 5.5, the first few Laguerre polynomials defined in the interval [-5, 20] are obtained from recurrence formula (3)are given below

$L_0(x) = 1, L_1(x) = 1 - x,$

$$L_2(x) = \frac{1}{2}(x^2 - 4x + 2), \quad L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$$

$$L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$$

$$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$$

$$L_6(x) = \frac{1}{720}(x^6 - 35x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720)$$

2.2 Chebyshev Polynomials:

The Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equation and denoted $T_n(x)$. They are used as an approximation to a least squares fit.The general form of the Chebyshev polynomials of n^{th} degree is defined by

$T_n(x) = \cos(n\theta)$ (4)

where $\cos(\theta) = x$. Using Scientific Workplace package 5.5, the first few Chebyshev polynomials defined in the interval [-1, 1] are obtained from recurrence formula (4)given below

$T_0(x) = 1, T_1(x) = x,$

$T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$, $T_5(x) = 16x^5 - 20x^3 + 5x$, $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

2.3 Legendre Polynomials

The Legendre polynomials are a well known family of orthogonal polynomials on the interval $[-1,1]$. They are solutions to the popular Legendre differential equation

$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ (5)

They are widely used because of their smooth properties in the approximation of functions. Equation (5) can be solved by series solution method (See Zill and Warren, 2013). The first few Legendre polynomials that can be obtained by solving (5) using the series solution method or by using the Rodriquez formula are:

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$,

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

2.4 Hermite Polynomials

The general form of the Hermite polynomials of n^{th} degree is defined by

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n e^{-x^2}}{dx^n}, \quad n = 0,1,2,3,\dots, \quad -\infty < x < \infty \qquad (6)$$

Below are some of the first few Hermite polynomials obtained from recurrence relation (6)

$H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$,

$H_4(x) = 16x^4 - 48x^2 + 12$, $H_5(x) = 35x^5 - 160x^3 + 120x$

$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

III. DERIVATION OF THE METHOD

An integral equation of the form;

$\int_{x_0}^{x} k(x,t)y(t)dt = f(x), \ l_0 \le x \le l_1$ (7)

where $\mathcal{L}(x,t)$ is the integral kernel (nucleus), $f(x)$ is a specified real valued function defined on $[l_0, l_1]$, and $y(x)$ is the function to be solved for is called a Volterra integral equation of the first kind where the unknown function $y(x)$ only appears inside the integral sign with $f(x)$ strictly satisfying $f(x_0) = 0$.

To find the approximate solution $\bar{y}(x)$ of (7), we shall assume that

$\bar{y}(x) \cong y(x) = \sum_{i=0}^{n} a_i H_i(x)$ (8)

where $H_i(x)$ is a Hermite polynomials of degree i defined in (1), n represent the number of Hermite polynomials and a_i's are the unknown constant parameters we need to determine. Now substituting (8) in (7) to obtain;

$\sum_{i=0}^{n} a_i \int_{x_0}^{x} k(x,t)H_i(x)dt = f(x), \ l_0 \le x \le$ (9)

In order to obtain the systems of linear equations that will enable us solve for the unknown coefficients in (8), we introduce the Galerkin's method by multiplying both sides of equation (9)

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by $H_q(x)$ and integrating the result with respect to x over the interval $[l_0, l_1]$ to obtain

$$\sum_{i=0}^{n} a_i \int_{l_0}^{l_1} \left(\int_{x_0}^{x} k(x,t) H_i(t) dt \right) H_q(x) dx = \int_{l_0}^{l_1} f(x) H_q(x) dx \,, q = 0,1, \dots, n \qquad (10)$$

Equation (10) can be put in matrix form as;

$$\sum_{i=0}^{n} a_i \,\Psi_{i,q} = \Omega_q \,, i, q = 0,1,2, \dots, n \qquad (11)$$

where $\Psi_{i,q} = \int_{l_0}^{l_1} \left[\int_{x_0}^{x} k(x,t) H_i(t) dt \right] H_q(x) dx$, $i, q = 0,1,2, \dots, n$ and $\Omega_q = \int_{l_0}^{l_1} f(x) H_q(x) dx$, $q = 0,1,2, \dots, n$

IV. NUMERICAL ILLUSTRATIONS

This section illustrates the above mentioned method using the following three examples of the first kind with regular kernels available in the existing literature (Rahman et al., (2012) and Farshid (2012)). The computations, associated with these examples are performed using the Scientific Workplace 5.5 Software.

4.1. Example 1: Consider an Abel's integral equation (VIE of first kind with weakly singular kernels) of the form (see, Rahman et al., (2012))

$$\int_0^x \frac{1}{\sqrt{(x-t)}} y(t) dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3), \quad 0 \le x \le 1$$

The exact solution is $y(x) = x^3 - x^2 + 1$ using the illustrated method in equation (8) and solving the linear system for $n = 1,2, \dots, 6$ we obtain the required value n for which the approximate solution converges to the exact solution as shown in Tables 1

4.2. Example 2

Consider the Volterra integral equation of the first kind (see, Farshid (2012)) given by,

$$\int_0^x e^{-x+t} y(t) dt = x, \quad 0 \le x \le 1$$

With the exact solution given as $y(x) = x + 1$, using the method illustrated method and solving the linear system for $n = 1,2, \dots, 6$ we obtain the required value n for which the approximate solution converges to the exact solution as shown in Tables 2

4.3. Example 3

Consider the Volterra integral equation of the first kind (see, Rahman et al., (2012)) given by,

$$\int_0^x 3^{x-t} y(t) dt = x, \quad 0 \le x \le 1$$

With exact solution $y(x) = 1 - x \log_e 3$, using the method illustrated in equation (8), solving the linear system for $n = 1,2, \dots, 6$ we obtain the required value of n for which the approximate solution converges to the exact solution as shown in Tables 3

Table 1. Value of n for which the approximate solution converges to the exact solution for problem 1

Polynomial basis function	Value of n for which $\bar{y}(x) = y(x)$
Bernoulli	$\bar{y}(x) \neq y(x)$ for all n
Chebyshev	$\bar{y}(x) = y(x)$ for $n = 3$
Hermite	$\bar{y}(x) = y(x)$ for $n = 3$
Laguerre	$\bar{y}(x) \neq y(x)$ for all n
Legendre	$\bar{y}(x) = y(x)$ for $n = 3$

Table 2. Value of n for which the approximate solution converges to the exact solution for problem 2

Polynomial basis function	Value of n for which $\bar{y}(x) = y(x)$
Bernoulli	$\bar{y}(x) = y(x)$ for $n = 1$
Chebyshev	$\bar{y}(x) = y(x)$ for $n = 1$
Hermite	$\bar{y}(x) = y(x)$ for $n = 1$
Laguerre	$\bar{y}(x) \neq y(x)$ for all n
Legendre	$\bar{y}(x) = y(x)$ for $n = 1$

Table 3. Value of n for which the approximate solution converges to the exact solution for problem 3

Polynomial basis function	Value of n for which $\bar{y}(x) = y(x)$
Bernoulli	$\bar{y}(x) = y(x)$ for $n = 1$
Chebyshev	$\bar{y}(x) = y(x)$ for $n = 1$
Hermite	$\bar{y}(x) = y(x)$ for $n = 1$
Laguerre	$\bar{y}(x) \neq y(x)$ for all n
Legendre	$\bar{y}(x) = y(x)$ for $n = 1$

V. CONCLUSION

In this paper, Volterra integral equations of the first kind are solved using five different polynomial basis functions. In all the cases, the approximate solution coincides with the exact solution at the same value of n, except in the case of Laguerre polynomials where there is no such value n for which the approximate solution and the exact solution coincides in all the examples considered. Thus the authors' conclude that the choice of the polynomial basis function does not affect the convergence of the approximate solution to the exact solution.

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