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A Study on the Non-homogeneous Ternary Quadratic Diophantine Equation

$$4\left(x^2+y^2\right)-7xy+x+y+1=31z^2$$

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ABSTRACT
The Non-homogeneous ternary quadratic Diophantine equation $4\left(x^2+y^2\right)-7xy+x+y+1=31z^2$ is studied for finding its non – zero distinct integer solutions.

KEY WORDS: Non-homogeneous, Ternary quadratic equation, Integral solutions .

I. INTRODUCTION

Ternary quadratic equations are rich in variety [1-4, 17-20].For an extensive review of sizable literature and various problems, one may refer [5-16]. In this communication, we consider yet another interesting Non- homogeneous ternary quadratic equation $4\left(x^2+y^2\right)-7xy+x+y+1=31z^2$ and obtain infinitely many non-trivial integral solutions.

II. METHOD OF ANALYSIS

Let x, y, z be any three non-zero distinct integers such that

$$4\left(x^2+y^2\right)-7xy+x+y+1=31z^2 \quad (1)$$

Substituting

$$\left.\begin{aligned} x&=u+v \\ y&=u-v \end{aligned}\right\} \quad (2)$$

in (1), we have

$$U^2+15v^2=31z^2 \quad (3)$$

where

$$u+1=U \quad (4)$$

(3) is solved through different methods for obtaining the values of U, v, z .In view of (4) and (2), the corresponding values of x, y are obtained.

The above process is illustrated below:

METHOD-1

(3) is written in the form of ratio as

$$\frac{U+4z}{(z-v)}=\frac{15(z+v)}{U-4z}=\frac{\alpha}{\beta},\ \beta\neq 0 \quad (5)$$

which is equivalent to the system of equations

$$U\beta+v\alpha+(4\beta-\alpha)z=0$$

$$-U\alpha+15v\beta+(4\alpha+15\beta)\,z=0$$

Employing the method of cross multiplication and simplifying, we have

$$U=4\alpha^2-60\beta^2+30\alpha\beta \quad (6)$$

$$v=\alpha^2-15\beta^2-8\alpha\beta \quad (7)$$

$$z=\alpha^2+15\beta^2 \quad (8)$$

Using (6) in (4) we have

$$u=4\alpha^2-60\beta^2+30\alpha\beta-1 \quad (9)$$

Using (7) and (9) in (2), we have

$$\left.\begin{aligned} x&=5\alpha^2-75\beta^2+22\alpha\beta-1 \\ y&=3\alpha^2-45\beta^2+38\alpha\beta-1 \end{aligned}\right\} \quad (10)$$

Thus (8) and (10) represent the non-zero distinct integer solution to (1).

NOTE:

In addition to (5), (3) is written in the form of ratio as below:

(i) $\dfrac{U+4z}{15(z-v)}=\dfrac{(z+v)}{U-4z}=\dfrac{\alpha}{\beta},\ \beta\neq 0$

(ii) $\dfrac{U+4z}{5(z-v)}=\dfrac{3(z+v)}{U-4z}=\dfrac{\alpha}{\beta},\ \beta\neq 0$

(iii) $\dfrac{U+4z}{3(z-v)}=\dfrac{5(z+v)}{U-4z}=\dfrac{\alpha}{\beta},\ \beta\neq 0$

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Following the procedure as above, the corresponding integer solutions to (1) thus obtained from each of the above cases are exhibited below:

Solutions from (i):

$$x = 75\alpha^2 - 5\beta^2 + 22\alpha\beta - 1$$

$$y = 45\alpha^2 - 3\beta^2 + 38\alpha\beta - 1$$

$$z = 15\alpha^2 + \beta^2$$

Solutions from (ii):

$$x = 25\alpha^2 - 15\beta^2 + 22\alpha\beta - 1$$

$$y = 15\alpha^2 - 9\beta^2 + 38\alpha\beta - 1$$

$$z = 5\alpha^2 + 3\beta^2$$

Solutions from (iii):

$$x = 15\alpha^2 - 25\beta^2 + 22\alpha\beta - 1$$

$$y = 9\alpha^2 - 15\beta^2 + 38\alpha\beta - 1$$

$$z = 3\alpha^2 + 5\beta^2$$

METHOD 2:

Introducing the linear transformations

$$z = X + 15T, \quad v = X + 31T, \quad U = 4W \quad (11)$$

in (3), it reduces to

$$X^2 = 465T^2 + W^2 \quad (12)$$

which is satisfied by

$$\left.\begin{aligned} X &= 465r^2 + s^2 \\ T &= 2rs \\ W &= 465r^2 - s^2 \end{aligned}\right\} \quad (13)$$

Using (13) in (11), we get

$$z = 465r^2 + s^2 + 30rs \quad (14)$$

$$v = 465r^2 + s^2 + 62rs \quad (15)$$

$$U = 4\left(465r^2 - s^2\right) \quad (16)$$

In view of (4), note that

$$u = 1860r^2 - 4s^2 - 1 \quad (17)$$

Using (15) and (17) in (2), we have

$$\left.\begin{aligned} x &= 2325r^2 - 3s^2 + 62rs - 1 \\ y &= 1395r^2 - 5s^2 - 62rs - 1 \end{aligned}\right\} \quad (18)$$

Thus (14) and (18) represent the non-zero distinct integer solutions to (1).

Further , (12) can be expressed as the system of double equations as shown in Table 1 below:

Table 1: System of double equations

System	1	2	3	4	5	6	7	8	9	10
$X+W$	465	T^2	$5T^2$	$15T^2$	$31T^2$	$155T^2$	465T	93T	31T	155T
$X-W$	T^2	465	93	31	15	3	T	5T	15T	3T

For simplicity and brevity, the integer solutions to (1) obtained on solving each of the above system of equations are exhibited in Table 2 below:

Table 2: Solutions

System	x	y	z
1	$-6k^2 + 56k + 1191$	$-10k^2 - 72k + 663$	$2k^2 + 32k + 248$
2	$10k^2 + 72k - 665$	$6k^2 - 56k - 1193$	$2k^2 + 32k + 248$
3	$50k^2 + 112k - 97$	$30k^2 - 32k - 257$	$10k^2 + 40k + 64$
4	$150k^2 + 212k + 21$	$90k^2 + 28k - 87$	$30k^2 + 60k + 38$
5	$310k^2 + 372k + 85$	$186k^2 + 124k - 23$	$62k^2 + 92k + 38$
6	$1550k^2 + 1612k + 413$	$930k^2 + 868k + 193$	$310k^2 + 340k + 94$
7	$1192T - 1$	$664T - 1$	$248T$
8	$256T - 1$	$96T - 1$	$64T$
9	$86T - 1$	$-22T - 1$	$38T$
10	$414T - 1$	$194T - 1$	$94T$

METHOD 3:

Write z as

$$z=\alpha^2+15\beta^2 \quad (19)$$

Also, 31 is written as

$$31=\left(4+i\sqrt{15}\right)\left(4-i\sqrt{15}\right) \quad (20)$$

Substituting (19) and (20) in (3) and employing the factorization method, define

$$\left(U+i\sqrt{15}v\right)=\left(4+i\sqrt{15}\right)\left(\alpha+i\sqrt{15}\beta\right)^2$$

On equating the real and imaginary parts, we have

$$U=4\alpha^2-60\beta^2-30\alpha\beta \quad (21)$$

$$v=\alpha^2-15\beta^2+8\alpha\beta \quad (22)$$

Using (21) in (4) we have

$$u=4\alpha^2-60\beta^2-30\alpha\beta-1 \quad (23)$$

Using (22) and (23) in (2) we have

$$\left.\begin{array}{l} x=5\alpha^2-75\beta^2-22\alpha\beta-1 \\ y=3\alpha^2-45\beta^2-38\alpha\beta-1 \end{array}\right\} \quad (24)$$

Thus (19) and (24) represent the non-zero distinct integer solutions to (1).

METHOD 4:

One may write (3) as

$$U^2+15v^2=31z^2*1 \quad (25)$$

Write 1 as

$$1=\frac{\left(1+i\sqrt{15}\right)\left(1-i\sqrt{15}\right)}{16} \quad (26)$$

Substituting (19), (20) and (26) in (25) and employing the factorization method, define

$$\left(U+i\sqrt{15}v\right)=\left(4+i\sqrt{15}\right)\left(\alpha+i\sqrt{15}\beta\right)^2*\frac{\left(1+i\sqrt{15}\right)}{4}$$

On equating the real and imaginary parts, we have

$$U=\frac{1}{4}\left(-11\alpha^2+165\beta^2-150\alpha\beta\right)$$

(27)

$$v=\frac{1}{4}\left(5\alpha^2-75\beta^2-22\alpha\beta\right) \quad (28)$$

Using (27) in (4) we have

$$u=\frac{1}{4}\left(-11\alpha^2+165\beta^2-150\alpha\beta-4\right)$$

(29)

Using (28) and (29) in (2) we have

$$\left.\begin{array}{l} x=\frac{1}{4}\left(-6\alpha^2+90\beta^2-172\alpha\beta-4\right) \\ y=\frac{1}{4}\left(-16\alpha^2+240\beta^2-128\alpha\beta-4\right) \end{array}\right\} \quad (30)$$

Thus (19) and (30) represent the non-zero distinct integer solutions to (1) when replacing α by 2α and β by 2β.

NOTE:

It is worth mentioning here that in addition to (26), 1 may be represented as below:

(i) $$1=\frac{\left(7+i\sqrt{15}\right)\left(7-i\sqrt{15}\right)}{64}$$

(ii) $$1=\frac{\left(7+i4\sqrt{15}\right)\left(7-i4\sqrt{15}\right)}{289}$$

(iii) $$1=\frac{\left(1+i8\sqrt{15}\right)\left(1-i8\sqrt{15}\right)}{961}$$

(iv) $$1=\frac{\left(7+i12\sqrt{15}\right)\left(7-i12\sqrt{15}\right)}{2209}$$

Following the procedure presented as above, for simplicity and brevity, we present below the integer solutions to (1) for (i) to (iv).

Solutions for (i):

$$x=4\left(3\alpha^2-45\beta^2-38\alpha\beta\right)-1$$
$$y=\alpha^2-15\beta^2-178\alpha\beta-1$$
$$z=4\left(\alpha^2+15\beta^2\right)$$

Solutions for (ii):

$$x=17\left(-9\alpha^2+135\beta^2-754\alpha\beta\right)-1$$
$$y=17\left(-55\alpha^2+825\beta^2-626\alpha\beta\right)-1$$
$$z=17^2\left(\alpha^2+15\beta^2\right)$$

Solutions for (iii):

$$x=31\left(-83\alpha^2+1245\beta^2-1222\alpha\beta\right)-1$$
$$y=31\left(-149\alpha^2+2235\beta^2-758\alpha\beta\right)-1$$
$$z=31^2\left(\alpha^2+15\beta^2\right)$$

Solutions for (iv):

$$x=47\left(-97\alpha^2+1455\beta^2-1954\alpha\beta\right)-1$$
$$y=47\left(-207\alpha^2+3105\beta^2-1346\alpha\beta\right)-1$$
$$z=47^2\left(\alpha^2+15\beta^2\right)$$

III. CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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