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A New Hybrid Block Method for Integrating Stiff IVP of ODEs

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ABSTRACT

A newhybrid variable step size of blockbackward differentiationformula (HVSSBBDF)for integrating stiff initial value problems of ordinary differential equation was introduced. The new scheme employed a variable step size technique. The proposed method canapproximatestwo solution values and two off-step values at each iteration. The stability analysishas been carried out.The method is found to satisfy the entire stability criteria, so the scheme is zero and A-stable capable for handling stiff IVPs. Different stable method would be obtains by varying the step size ratio appropriately in the formula. Existingstiff IVPs are solved using the proposed scheme, the results are found to validate the performances of the new scheme in terms of accuracy of the scale error andminimum execution time for most of the problem considered in the research. Hence, the proposed new scheme is recommended for the solutions to stiff IVPs of ODEs.

Keywords: A – stable,block method, stiff ODEs, variable step, zero stable

I. INTRODUCTION

An interesting aspect of mathematics is the ability to transforms real-life problemsinto mathematical models and then solve with a relevant methods. Most of the scientific and engineering problems are transform into a differential equations. Such equations can either be partial or ordinary differential equations. Obtaining solutions to those differential equations are the ultimate goals of the scholars that modeled the equations. Unfortunately, often times the modeled equations turn to be stiff in the solution, a phenomena that comprises transient and steady state in the solution. Stiff problem usually deviates from been solved analytically due to its complexities and other phenomena which is found within its solution, the transient and steady state components found in its solution make explicit method difficult to handle with appreciated results. Hence, preferences are always channels to numerical methods that would solve any sort of stiff IVP of ODEs. The ultimate goal is to get a method with a solution that has absolutely minimum scale error and computational time. Backward differentiation formula came to be developed by Curtiss and Hirschfield [1], several extensions of [1] was carrryout by many scholars incuding Cash [2-3], the block aspect of [1] was formulated by [4]. Different scholars work tremendeously in devicing a BBDF method that can handle stiff IVPS with minimum error and computational time, these can be found in [5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15], [16] & [17]. Several researchers also contributed immensely in solving numerical problems related to real life aspect such include [18], [19], [20], [21], [22],[23],[24]& [25].

This study considers deriving avariable step size block backward differentiationaspect of [16]. It is of the form

$$\sum_{j=0}^{2} \alpha_{j,i} y_{n+j-2} + \sum_{j=0}^{3} \alpha_{j+3,i} y_{n+(j+1/2)} = h\beta_i f_{n+k} \qquad k = \frac{1}{2}, 1, \frac{3}{2}, 2 \qquad (1)$$

The propose scheme is archived by maintaining the same order whilemodifying (1) to consider the step size variable r, to came-up with another hybrid method with variable step size strategy of the form

$$\sum_{j=0}^{2} \alpha_{j,i,r} y_{n+j-2} + \sum_{j=0}^{3} \alpha_{j+3,i} y_{n+(j+1/2)} = h\beta_{k+1i} f_{n+k} \qquad k = \frac{1}{2}, 1, \frac{3}{2}, 2 \qquad (2)$$

The proposed method approximates two solution valuesof y_{n+1}, y_{n+2} and two off-step values of $y_{n+\frac{1}{2}}$, $y_{n+\frac{3}{2}}$at a time simultaneously, per integration step for a stiff ODE of the following form

$$\left.\begin{array}{c} y' = f(x, \hat{Y}), \ \hat{Y}(a) = \tau, \ a \le x \le b \\ \hat{Y} = (y_1, y_2, y_3, \dots \dots \dots y_n), \eta \ \bar{\tau} = (\tau\eta_1, \tau\eta_2, \tau\eta_3, \dots, \tau\eta_n) \end{array}\right\} \quad (3)$$

II. METHODOLOGY

2.1 Formulation of the Proposed Method

In this section, two approximate solution values y_{n+1} and y_{n+2} with step size h, and two off-step points $y_{n+\frac{1}{2}}$ and $y_{n+\frac{3}{2}}$ which are chosen at the points where the step size is halved are formulated in a block simultaneously. The formulae are computed using two back values y_n and y_{n-1} with step size h.

The proposed method(2) is used in this formulation, where k and i have the same value. The formula (2) is derived using Taylor's series expansion about x_n

2.1.1 Definition: According to [17], the linear operator $L_i$associated with first, second, third and fourth point of the method with off-step point method isdefined as follows:

Consider the following value of $k \,\&\, i's$ value in (3) for the cases below:

$$L_i[y(x_n), h]: \alpha_{0,i}y_{n-2} + \alpha_{1,i}y_{n-1} + \alpha_{2,i}y_n + \alpha_{3,i}y_{n+\frac{1}{2}} + \alpha_{4,i}y_{n+1} + \alpha_{5,i}y_{n+\frac{3}{2}} + \alpha_{6,i}y_{n+2} \ - h\beta_{k+1,i}f_{n+k} = 0 \quad (3)$$

CASE 1: $k = i = \frac{1}{2}$

$$L_{\frac{1}{2}}[y(x_n), h]: \alpha_{0,\frac{1}{2}}y_{n-2} + \alpha_{1,\frac{1}{2}}y_{n-1} + \alpha_{2,\frac{1}{2}}y_n + \alpha_{3,\frac{1}{2}}y_{n+\frac{1}{2}} + \alpha_{4,\frac{1}{2}}y_{n+1} + \alpha_{5,\frac{1}{2}}y_{n+\frac{3}{2}} + \alpha_{6,\frac{1}{2}}y_{n+2} \ - h\beta_{\frac{3}{2},\frac{1}{2}}\left[f\left(x_n + \frac{1}{2}h\right)\right]$$

(4)

CASE 2: $k = i = 1$

$$L_1[y(x_n), h]: \alpha_{0,1}y_{n-2} + \alpha_{1,1}y_{n-1} + \alpha_{2,1}y_n + \alpha_{3,1}y_{n+\frac{1}{2}} + \alpha_{4,1}y_{n+1} + \alpha_{5,1}y_{n+\frac{3}{2}} + \alpha_{6,1}y_{n+2} \ - h\beta_{2,1}[f(x_n + h)] = 0 \quad (5)$$

CASE 3: $k = i = \frac{3}{2}$

$$L_{\frac{3}{2}}[y(x_n), h]: \alpha_{0,\frac{3}{2}}y_{n-2} + \alpha_{1,\frac{3}{2}}y_{n-1} + \alpha_{2,\frac{3}{2}}y_n + \alpha_{3,\frac{3}{2}}y_{n+\frac{1}{2}} + \alpha_{4,\frac{3}{2}}y_{n+1} + \alpha_{5,\frac{3}{2}}y_{n+\frac{3}{2}} + \alpha_{6,\frac{3}{2}}y_{n+2} \ - h\beta_{\frac{5}{2},\frac{3}{2}}\left[f\left(x_n + \right.\right.$$

32h=0 (6)

CASE 4: $k = i = 2$

$$L_2[y(x_n), h]: \alpha_{0,2}y_{n-2} + \alpha_{1,2}y_{n-1} + \alpha_{2,2}y_n + \alpha_{3,2}y_{n+\frac{1}{2}} + \alpha_{4,2}y_{n+1} + \alpha_{5,2}y_{n+\frac{3}{2}} + \alpha_{6,2}y_{n+2} \ - h\beta_{3,2}[f(x_n +$$

2h=0 (7)

From cases 1,2,3& 4, we have the following linear operators

$$\left.\begin{array}{l} L_{\frac{1}{2}}[y(x_n),h]: \alpha_{0,\frac{1}{2}}y_{n-2} + \alpha_{1,\frac{1}{2}}y_{n-1} + \alpha_{2,\frac{1}{2}}y_n + \alpha_{3,\frac{1}{2}}y_{n+\frac{1}{2}} + \alpha_{4,\frac{1}{2}}y_{n+1} + \alpha_{5,\frac{1}{2}}y_{n+\frac{3}{2}} + \alpha_{6,\frac{1}{2}}y_{n+2} \ - h\beta_{\frac{3}{2},\frac{1}{2}}\left[f\left(x_n + \frac{1}{2}h\right)\right] = 0 \\ L_1[y(x_n),h]: \alpha_{0,1}y_{n-2} + \alpha_{1,1}y_{n-1} + \alpha_{2,1}y_n + \alpha_{3,1}y_{n+\frac{1}{2}} + \alpha_{4,1}y_{n+1} + \alpha_{5,1}y_{n+\frac{3}{2}} + \alpha_{6,1}y_{n+2} \ - h\beta_{2,1}[f(x_n + h)] = 0 \\ L_{\frac{3}{2}}[y(x_n),h]: \alpha_{0,\frac{3}{2}}y_{n-2} + \alpha_{1,\frac{3}{2}}y_{n-1} + \alpha_{2,\frac{3}{2}}y_n + \alpha_{3,\frac{3}{2}}y_{n+\frac{1}{2}} + \alpha_{4,\frac{3}{2}}y_{n+1} + \alpha_{5,\frac{3}{2}}y_{n+\frac{3}{2}} + \alpha_{6,\frac{3}{2}}y_{n+2} \ - h\beta_{\frac{5}{2},\frac{3}{2}}\left[f\left(x_n + \frac{3}{2}h\right)\right] = 0 \\ L_2[y(x_n),h]: \alpha_{0,2}y_{n-2} + \alpha_{1,2}y_{n-1} + \alpha_{2,2}y_n + \alpha_{3,2}y_{n+\frac{1}{2}} + \alpha_{4,2}y_{n+1} + \alpha_{5,2}y_{n+\frac{3}{2}} + \alpha_{6,2}y_{n+2} \ - h\beta_{3,2}[f(x_n + 2h)] = 0 \end{array}\right\}$$

Expanding$(x_n - 2h),(x_n - h),\ y(x_n),y(x_n + \frac{1}{2}h),y(x_n + h),y(x_n + \frac{3}{2}h),y(x_n + 2h),f\left(x_n + \frac{1}{2}h\right),f\left(x_n + \right.$

Expanding$(x_n - 2h),(x_n - h),\ y(x_n),y(x_n + \frac{1}{2}h),y(x_n + h),y(x_n + \frac{3}{2}h),y(x_n + 2h),f\left(x_n + \frac{1}{2}h\right),f\left(x_n + \right.$ 32h,fxn+h,fxn+2hin (4) with a Taylor's series expansion about xnand collect the like terms and rearrange, we have

$$\left.\begin{array}{r} C_{0,\frac{1}{2}}y(x_n) + C_{1,\frac{1}{2}}hy'(x_n) + C_{2,\frac{1}{2}}h^2y''(x_n) + C_{3,\frac{1}{2}}h^3y''(x_n) + C_{4,\frac{1}{2}}h^4y''(x_n) + \cdots = 0 \\ C_{0,1}y(x_n) + C_{1,1}hy'(x_n) + C_{2,1}h^2y''(x_n) + C_{3,1}h^3y'''(x_n) + C_{4,1}h^4y'''(x_n) + \cdots = 0 \\ C_{0,\frac{3}{2}}y(x_n) + C_{1\frac{3}{2},}hy'(x_n) + C_{2,\frac{3}{2}}h^2y''(x_n) + C_{3,\frac{3}{2}}h^3y'''(x_n) + C_{4,\frac{3}{2}}h^4y'^V(x_n) + \cdots = 0 \\ C_{0,2}y(x_n) + C_{1,2}hy'(x_n) + C_{2,2}h^2y''(x_n) + C_{3,2}h^3y'''(x_n) + C_{4,2}h^4y'^V(x_n) + \cdots = 0 \end{array}\right\} (8)$$

where (8) is evaluated as in (9),(10),(11) & (12),for case 1,2,3 & 4 respectively as follows

$$\left.\begin{aligned}
C_{0,\frac{1}{2}} &= \alpha_{0,\frac{1}{2}} + \alpha_{1,\frac{1}{2}} + \alpha_{2,\frac{1}{2}} + \alpha_{4,\frac{1}{2}} + \alpha_{5,\frac{1}{2}} + \alpha_{6,\frac{1}{2}} = -1\\
C_{1,\frac{1}{2}} &= -2r\alpha_{0,\frac{1}{2}} - r\alpha_{1,\frac{1}{2}} + \alpha_{4,\frac{1}{2}} + \frac{3}{2}\alpha_{5,\frac{1}{2}} + 2\alpha_{6,\frac{1}{2}} - \beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{2}\\
C_{2,\frac{1}{2}} &= 2r^2\alpha_{0,\frac{1}{2}} + \frac{1}{2}r^2\alpha_{1,\frac{1}{2}} + \frac{1}{2}\alpha_{4,\frac{1}{2}} + \frac{9}{8}\alpha_{5,\frac{1}{2}} + 2\alpha_{6,\frac{1}{2}} - \frac{1}{2}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{8}\\
C_{3,\frac{1}{2}} &= -\frac{4}{3}r^3\alpha_{0,\frac{1}{2}} - \frac{1}{6}r^3\alpha_{1,\frac{1}{2}} + \frac{1}{6}\alpha_{4,\frac{1}{2}} + \frac{27}{48}\alpha_{5,2} + \frac{4}{3}\alpha_{6,\frac{1}{2}} - \frac{1}{8}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{48}\\
C_{4,\frac{1}{2}} &= \frac{2}{3}r^4\alpha_{0,\frac{1}{2}} + \frac{1}{24}r^4\alpha_{1,\frac{1}{2}} + \frac{1}{24}\alpha_{4,\frac{1}{2}} + \frac{81}{384}\alpha_{5,\frac{1}{2}} + \frac{2}{3}\alpha_{6,\frac{1}{2}} - \frac{1}{48}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{384}\\
C_{5,\frac{1}{2}} &= -\frac{4}{15}r^5\alpha_{0,\frac{1}{2}} - \frac{1}{120}r^5\alpha_{1,\frac{1}{2}} + \frac{1}{120}\alpha_{4,\frac{1}{2}} + \frac{243}{3840}\alpha_{5,\frac{1}{2}} + \frac{4}{15}\alpha_{6,\frac{1}{2}} - \frac{1}{384}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{3840}\\
C_{6,\frac{1}{2}} &= \frac{4}{45}r^6\alpha_{0,\frac{1}{2}} + \frac{1}{720}r^6\alpha_{1,\frac{1}{2}} + \frac{1}{720}\alpha_{4,\frac{1}{2}} + \frac{729}{46080}\alpha_{5,\frac{1}{2}} + \frac{4}{45}\alpha_{6,\frac{1}{2}} - \frac{1}{3840}\beta_{\frac{3}{2},\frac{1}{2}} = -\frac{1}{46080}
\end{aligned}\right\}(9)$$

$$\left.\begin{aligned}
C_{0,1} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{4,1} + \alpha_{5,1} + \alpha_{6,1} = -1\\
C_{1,1} &= -2r\alpha_{0,1} - r\alpha_{1,1} + \frac{1}{2}\alpha_{4,1} + \frac{3}{2}\alpha_{5,1} + 2\alpha_{6,1} - \beta_{2,1} = -1\\
C_{2,1} &= 2r^2\alpha_{0,1} + \frac{1}{2}r^2\alpha_{1,1} + \frac{1}{8}\alpha_{4,1} + \frac{9}{8}\alpha_{5,1} + 2\alpha_{6,1} - 2\beta_{2,1} = -\frac{1}{2}\\
C_{3,1} &= -\frac{4}{3}r^3\alpha_{0,1} - \frac{1}{6}r^3\alpha_{1,1} + \frac{1}{48}\alpha_{4,1} + \frac{27}{48}\alpha_{5,2} + \frac{4}{3}\alpha_{6,1} - 2\beta_{2,1} = -\frac{1}{6}\\
C_{4,1} &= \frac{2}{3}r^4\alpha_{0,1} + \frac{1}{24}r^4\alpha_{1,1} + \frac{1}{384}\alpha_{4,1} + \frac{81}{384}\alpha_{5,1} + \frac{2}{3}\alpha_{6,1} - \frac{4}{3}\beta_{2,1} = -\frac{1}{24}\\
C_{5,1} &= -\frac{4}{15}r^5\alpha_{0,1} - \frac{1}{120}r^5\alpha_{1,1} + \frac{1}{3840}\alpha_{4,1} + \frac{243}{3840}\alpha_{5,1} + \frac{4}{15}\alpha_{6,1} - \frac{2}{3}\beta_{2,1} = -\frac{1}{120}\\
C_{6,1} &= \frac{4}{45}r^6\alpha_{0,1} + \frac{1}{720}r^6\alpha_{1,1} + \frac{1}{46080}\alpha_{4,1} + \frac{729}{46080}\alpha_{5,1} + \frac{4}{45}\alpha_{6,1} - \frac{4}{15}\beta_{2,1} = -\frac{1}{720}
\end{aligned}\right\}(10)$$

$$\left.\begin{aligned}
C_{0,\frac{3}{2}} &= \alpha_{0,\frac{3}{2}} + \alpha_{1,\frac{3}{2}} + \alpha_{2,\frac{3}{2}} + \alpha_{4,\frac{3}{2}} + \alpha_{5,\frac{3}{2}} + \alpha_{6,\frac{3}{2}} = -1\\
C_{1,\frac{3}{2}} &= -2r\alpha_{0,\frac{3}{2}} - r\alpha_{1,\frac{3}{2}} + \frac{1}{2}\alpha_{4,\frac{3}{2}} + \alpha_{5,\frac{3}{2}} + 2\alpha_{6,\frac{3}{2}} - \beta_{\frac{5}{2},\frac{3}{2}} = -\frac{3}{2}\\
C_{2,\frac{1}{2}} &= 2r^2\alpha_{0,\frac{1}{2}} + \frac{1}{2}r^2\alpha_{1,\frac{1}{2}} + \frac{1}{8}\alpha_{4,\frac{1}{2}} + \frac{1}{2}\alpha_{5,\frac{1}{2}} + 2\alpha_{6,\frac{1}{2}} - \frac{3}{2}\beta_{\frac{5}{2},\frac{3}{2}} = -\frac{9}{8}\\
C_{3,\frac{1}{2}} &= -\frac{4}{3}r^3\alpha_{0,\frac{1}{2}} - \frac{1}{6}r^3\alpha_{1,\frac{1}{2}} + \frac{1}{48}\alpha_{4,\frac{1}{2}} + \frac{1}{6}\alpha_{5,2} + \frac{4}{3}\alpha_{6,\frac{1}{2}} - \frac{9}{8}\beta_{\frac{5}{2},\frac{3}{2}} = -\frac{27}{48}\\
C_{4,\frac{1}{2}} &= \frac{2}{3}r^4\alpha_{0,\frac{1}{2}} + \frac{1}{24}r^4\alpha_{1,\frac{1}{2}} + \frac{1}{384}\alpha_{4,\frac{1}{2}} + \frac{1}{24}\alpha_{5,\frac{1}{2}} + \frac{2}{3}\alpha_{6,\frac{1}{2}} - \frac{27}{48}\beta_{\frac{5}{2},\frac{3}{2}} = -\frac{81}{384}\\
C_{5,\frac{1}{2}} &= -\frac{4}{15}r^5\alpha_{0,\frac{1}{2}} - \frac{1}{120}r^5\alpha_{1,\frac{1}{2}} + \frac{1}{3840}\alpha_{4,\frac{1}{2}} + \frac{1}{120}\alpha_{5,\frac{1}{2}} + \frac{4}{15}\alpha_{6,\frac{1}{2}} - \frac{81}{384}\beta_{\frac{5}{2},\frac{3}{2}} = -\frac{243}{3840}\\
C_{6,\frac{1}{2}} &= \frac{4}{45}r^6\alpha_{0,\frac{1}{2}} + \frac{1}{720}r^6\alpha_{1,\frac{1}{2}} + \frac{1}{46080}\alpha_{4,\frac{1}{2}} + \frac{1}{720}\alpha_{5,\frac{1}{2}} + \frac{4}{45}\alpha_{6,\frac{1}{2}} - \frac{243}{3840}\beta_{\frac{5}{2},\frac{3}{2}} = -\frac{720}{46080}
\end{aligned}\right\}(11)$$

&

$$\left.\begin{aligned}
C_{0,2} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{4,1} + \alpha_{5,1} + \alpha_{6,1} = -1\\
C_{1,2} &= -2r\alpha_{0,1} - r\alpha_{1,1} + \frac{1}{2}\alpha_{4,1} + \alpha_{5,1} + \frac{3}{2}\alpha_{6,1} - \beta_{3,2} = -2\\
C_{2,2} &= 2r^2\alpha_{0,1} + \frac{1}{2}r^2\alpha_{1,1} + \frac{1}{8}\alpha_{4,1} + \frac{1}{2}\alpha_{5,1} + \frac{9}{8}\alpha_{6,1} - 2\beta_{3,2} = -2\\
C_{3,2} &= -\frac{4}{3}r^3\alpha_{0,1} - \frac{1}{6}r^3\alpha_{1,1} + \frac{1}{48}\alpha_{4,1} + \frac{1}{6}\alpha_{5,2} + \frac{27}{48}\alpha_{6,1} - 2\beta_{3,2} = -\frac{4}{3}\\
C_{4,2} &= \frac{2}{3}r^4\alpha_{0,1} + \frac{1}{24}r^4\alpha_{1,1} + \frac{1}{384}\alpha_{4,1} + \frac{1}{24}\alpha_{5,1} + \frac{81}{384}\alpha_{6,1} - \frac{4}{3}\beta_{3,2} = -\frac{2}{3}\\
C_{5,2} &= -\frac{4}{15}r^5\alpha_{0,1} - \frac{1}{120}r^5\alpha_{1,1} + \frac{1}{3840}\alpha_{4,1} + \frac{1}{120}\alpha_{5,1} + \frac{243}{3840}\alpha_{6,1} - \frac{2}{3}\beta_{3,2} = -\frac{4}{15}\\
C_{6,2} &= \frac{4}{45}r^6\alpha_{0,1} + \frac{1}{720}r^6\alpha_{1,1} + \frac{1}{46080}\alpha_{4,1} + \frac{1}{720}\alpha_{5,1} + \frac{729}{46080}\alpha_{6,1} - \frac{4}{15}\beta_{3,2} = -\frac{4}{45}
\end{aligned}\right\} \qquad (12)$$

Normalizing the coefficients $\alpha_{3,\frac{1}{2}}$, $\alpha_{4,1}$, $\alpha_{5,\frac{3}{2}}$&$\alpha_{6,2}$of $y_{n+\frac{1}{2}}$, y_{n+1} , $y_{n+\frac{3}{2}}$&y_{n+2} respectively to 1. Solving equation (9), (10), (11) & (12) with the aids of Maple Software for the values of $\alpha_{j,i}$ and $\beta_{j,i}$ and Substituting the values in (4-7) gives the first, second, third & fourth point as

$$y_{n+\frac{1}{2}} = \frac{9(2r+1)}{16(4r+3)(r+1)r^2(40r^2-6r-7)}y_{n-2} + \frac{9(16r^2+8r+1)}{4r^2(40r^4+114r^3+55r^2-33r-14)(2r+3)}y_{n-1} - \frac{3(64r^4+9r^3+5r^2+12r+1)}{16r^2(40r^2-6r-7)}y_n + \frac{9(32r^2+32r^2+10r+1)}{4(r+1)(40r^2-6r-7)}y_{n+1} - \frac{3(64r^4+96r^3+52r^2+12r+1)}{(8r^2+18r+9)(40r^2-6r-7)}y_{n+\frac{3}{2}} + \frac{64r^4+96r^3+52r^2+12r+1}{16(40r^4+114r^3+55r^2-33r-14)}y_{n+2} - \frac{3(8r^2+6r+1)}{40r^2-6r-7}hf_{n+\frac{1}{2}} \quad (10)$$

$$y_{n+1} = \frac{1}{8(r+1)(4r+3)(4r+1)}y_{n-2} + \frac{1}{r^2(r^2+4r+4)(2r+3)}y_{n-1} - \frac{2r^2+3r+1}{24r^2}y_n + \frac{16(2r^2+3r+1)}{9(r+1)}y_{n+\frac{1}{2}} + \frac{16(2r^2+3r+1)}{3(8r^2+18r+9)}y_{n+\frac{3}{2}} - \frac{50r^3+193r^2+208r+62}{72(r^3+5r^2+8r+4)}y_{n+2} + \frac{2r+1}{12(r+2)}hf_{n+1} \quad (11)$$

$$y_{n+\frac{3}{2}} = \frac{9(4r^2+12r+9)}{16(r+1)(r+1)(80r^3+292r^2+288r+81)(4r+1)r^2}y_{n-2} - \frac{9(16r^2+24r+9)}{4r^2(r+1)(r+2)(80r^3+292r^2+288r+81)}y_{n-1} + \frac{64r^4+288r^3+468r^2+324r+81}{16r^2(40r^2+126r+81)}y_n - \frac{3(64r^4+288r^3+468r^2+324r+81)}{(8r^2+16r+1)(40r^2+126r+81)}y_{n+\frac{1}{2}} + \frac{9(64r^4+288r^3+468r^2+324r+81)}{4(80r^4+372r^3+580r^2+369r+81)}y_{n+1} + \frac{3(64r^4+288r^3+468r^2+324r+81)}{16(r^2+3r+2)(40r^2+126r+81)}y_{n+2} + \frac{3(8r^2+18r+9)}{40r^2+126r+51}hf_{n+\frac{3}{2}} \quad (12)$$

$$y_{n+2} = \frac{9(r^2+4r+4)}{(100r^3+411r^2+500r+186)(8r^2+6r+1)}y_{n-2} + \frac{72(r+1)}{r^2(50r^3+243r^2+376r+186)(2r+1)}y_{n-1} - \frac{3(r^4+6r^3+13r^2+12r+4)}{r^2(25r^2+84r+62)}y_n + \frac{128(r^4+6r^3+13r^2+12r+4)}{(8r^2+6r+1)(25r^2+84r+62)}y_{n+\frac{1}{2}} - \frac{72(r^3+5r^2+8r+4)}{(2r+1)(25r^2+84r+62)}y_{n+1} + \frac{384(r^4+6r^3+13r^2+12r+4)}{200r^4+1122r^3+2233r^2+1872r+62}y_{n+\frac{3}{2}} + \frac{6(r^2+3r+2)}{25r^2+84r+62}hf_{n+2} \quad (13)$$

Hence, (10) - (13) is called a new hybrid block method of order 6(HVSSBBDF).
From the proposed scheme, different stable methods can be obtained by carefully varying the value of the step size ratio.

Table 2.1: Variable Step Size Ratios with the StableMethods obtained

Step size ratio (r)	Formulae (HVSSBBDF)
$r = 1$	$y_{n+\frac{1}{2}} = -\frac{1}{244}y_{n-2} + \frac{5}{72}y_{n-1} - \frac{25}{16}y_n + \frac{25}{8}y_{n+1} - \frac{5}{7}y_{n+\frac{3}{2}} + \frac{25}{288}y_{n+2} - \frac{5}{3}hf_{n+\frac{1}{2}}$ $y_{n+1} = -\frac{1}{560}y_{n-2} + \frac{1}{45}y_{n-1} - \frac{1}{4}y_n + \frac{32}{45}y_{n+\frac{1}{2}} + \frac{32}{35}y_{n+\frac{3}{2}} - \frac{19}{48}y_{n+2} + \frac{1}{12}hf_{n+1}$ $y_{n+\frac{3}{2}} = \frac{17}{7904}y_{n-2} - \frac{49}{1976}y_{n-1} + \frac{1225}{3952}y_n - \frac{245}{247}y_{n+\frac{1}{2}} + \frac{3675}{1976}y_{n+1} - \frac{1225}{7904}y_{n+2} + \frac{104}{247}hf_{n+\frac{3}{2}}$ $y_{n+2} = -\frac{3}{665}y_{n-2} + \frac{16}{285}y_{n-1} - \frac{12}{19}y_n + \frac{512}{285}y_{n+\frac{1}{2}} - \frac{48}{19}y_{n+1} + \frac{1536}{665}y_{n+\frac{3}{2}} + \frac{14}{19}hf_{n+2}$
$r = 2$	$y_{n+\frac{1}{2}} = -\frac{5}{33088}y_{n-2} + \frac{81}{21056}y_{n-1} - \frac{2025}{3008}y_n + \frac{405}{188}y_{n+1} - \frac{2025}{3619}y_{n+\frac{3}{2}} + \frac{225}{3008}y_{n+2} - \frac{45}{47}hf_{n+\frac{1}{2}}$ $y_{n+1} = -\frac{1}{9504}y_{n-2} + \frac{1}{448}y_{n-1} - \frac{5}{32}y_n + \frac{16}{27}y_{n+\frac{1}{2}} + \frac{80}{77}y_{n+\frac{3}{2}} - \frac{275}{576}y_{n+2} + \frac{5}{48}hf_{n+1}$ $y_{n+\frac{3}{2}} = \frac{49}{473280}y_{n-2} - \frac{363}{157760}y_{n-1} + \frac{5929}{31552}y_n - \frac{5929}{7395}y_{n+\frac{1}{2}} + \frac{17787}{9860}y_{n+1} - \frac{5929}{31552}y_{n+2} + \frac{231}{493}hf_{n+\frac{3}{2}}$

	$y_{n+2} = -\frac{2}{9075}y_{n-2} + \frac{9}{1925}y_{n-1} - \frac{18}{55}y_n + \frac{1024}{825}y_{n+\frac{1}{2}} - \frac{576}{275}y_{n+1} + \frac{9216}{4235}y_{n+\frac{3}{2}} + \frac{12}{55}hf_{n+2}$
$r = \frac{4}{5}$	$y_{n+\frac{1}{2}} = -\frac{8125}{547584}y_{n-2} + \frac{13125}{67712}y_{n-1} - \frac{74529}{29440}y_n + \frac{1911}{460}y_{n+1} - \frac{74529}{81995}y_{n+\frac{3}{2}} + \frac{1183}{11040}y_{n+2} - \frac{273}{115}hf_{n+\frac{1}{2}}$ $y_{n+1} = -\frac{3125}{749952}y_{n-2} + \frac{3125}{72128}y_{n-1} - \frac{39}{128}y_n + \frac{16}{21}y_{n+\frac{1}{2}} + \frac{624}{713}y_{n+\frac{3}{2}} - \frac{7865}{21168}y_{n+2} + \frac{13}{168}hf_{n+1}$ $y_{n+\frac{3}{2}} = \frac{330625}{72473856}y_{n-2} - \frac{600625}{12078976}y_{n-1} + \frac{508369}{1327360}y_n - \frac{508369}{471835}y_{n+\frac{1}{2}} + \frac{508369}{269620}y_{n+1} - \frac{508369}{3484320}y_{n+2} + \frac{2139}{5185}hf_{n+\frac{3}{2}}$ $y_{n+2} = -\frac{4375}{390104}y_{n-2} + \frac{16875}{144716}y_{n-1} - \frac{3969}{4840}y_n + \frac{16128}{7865}y_{n+\frac{1}{2}} - \frac{21168}{7865}y_{n+1} + \frac{1016064}{431365}y_{n+\frac{3}{2}} + \frac{126}{605}hf_{n+2}$

III. ANALYSIS OF THE PROPOSED METHOD

In this part, order and stability properties of the proposedmethod (10-13) will be analysed.

3.1 Order of the Method

The order of the method (10-13) and its associated linear operator is given by

$$L[y(x);\ h] = \sum_{j=0}^{11}[C_j\, y(x + jh)] - h\sum_{j=0}^{11}[D_j\, y'(x + jh)] \qquad (14)$$

where C_j , D_j are constant coefficient matrices and p is unique integer s.t.

$E_s = 0,\ s = 0,1, \dots\ p\ \ and\ E_{p+1} \neq 0, where\ the\ \ E_s\ are\ constant$ Matrices

$for\ r = 1$, we have

$$E_0 = \sum_{J=0}^{11} C_J = 0$$

$$E_1 = \sum_{J=0}^{11}[jC_j - 2D_j] = 0$$

$$E_2 = \sum_{J=0}^{11}\left[\frac{1}{2!}j^2C_j - 2jD_j\right] = 0$$

$$E_3 = \sum_{J=0}^{11}\left[\frac{1}{3!}j^3C_j - 2\frac{1}{2!}j^2D_j\right] = 0$$

$$E_4 = \sum_{J=0}^{11}\left[\frac{1}{4!}j^4C_j - 2\frac{1}{3!}j^3D_j\right] = 0$$

$$E_5 = \sum_{J=0}^{11}\left[\frac{1}{5!}j^5C_j - 2\frac{1}{4!}j^4D_j\right] = 0$$

$$E_6 = \sum_{J=0}^{11}\left[\frac{1}{6!}j^6C_j - 2\frac{1}{5!}j^5D_j\right] = 0$$

$$E_7 = \sum_{J=0}^{11}\left[\frac{1}{7!}j^7C_j - 2\frac{1}{6!}j^6D_j\right] = \begin{bmatrix} -19280011/9072 \\ 12256367/3150 \\ 3411415/1872 \\ -1204057/855 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (15)$$

Similarly, other methods would be generated using same procedure.

Table 3.1: All the Selected Step Size Ratio r

Step Size Ratio (r)	**Order**	**Error Constants**
$r = 1$	6	$[-19280011/9072 \quad 12256367/3150 \quad 3411415/1872 \quad -1204057/855]^T$
$r = 2$	6	[1050.7385 4.132.8158 1683.0258 3059.7655]^T
$r = \frac{4}{5}$	6	[3526.0365 333.8686 -11455673.6279 -1600.8544]^T

3.2 Zero Stability Analysis of the ProposedMethod

In this section, we investigate the zero and A- Stability property of the proposed method.

3.2.1 Definition: According to [17], a linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple.

3.2.2 Definition: According to [17],a linear multistep method is said to be an A-stable method if its stability region covers the entire negative half-plane.

The characteristic polynomial for $r = 1$ can be obtain with relation

$$det(At^2 - Bt - C) =$$

$$-\frac{5432344}{633555}t^8 + \frac{199656}{23465}t^7 + \frac{1544}{23465}t^6 - \frac{56}{633555}t^5 + \frac{2305267}{1783340}t^8h - \frac{4586687}{1783340}t^8h^2 \quad (16)$$
$$+ \frac{162555503}{26750100}t^7h + \frac{8991}{9386}t^8h^3 + \frac{6991}{2964}t^7h^2 - \frac{630}{4693}t^8h^4 + \frac{1869}{4693}t^7h^3$$
$$- \frac{21533}{1971060}t^6h + \frac{297}{93860}t^6h^2 + \frac{1727}{29565900}t^5h + \frac{27}{9386}t^6h^3 + \frac{1}{93860}t^5h^2$$

For $r = 2$

$$det(At^2 - Bt - C) = \frac{586800}{254881}t^8 - \frac{2283084}{1274405}t^7 - \frac{2489}{3823215}t^6 + \frac{1}{19116075}t^5 - \frac{11091911}{7646430}t^8h\text{ʎ} + \frac{38424}{115855}t^8(h\text{ʎ})^2 - \frac{2887809}{2548810}t^7h\text{ʎ} + \frac{635055}{4078096}t^8(h\text{ʎ})^3 + \frac{82674}{127405}t^2(h\text{ʎ})^2 - \frac{945}{92684}t^8(h\text{ʎ})^4 + \frac{5589}{370736}t^7(h\text{ʎ})^3 + \frac{949}{1529286}t^6h\text{ʎ} + \frac{26}{1274405}t^6(h\text{ʎ})^2 - \frac{1}{12744050}t^5h\text{ʎ} + \frac{21}{2039048}t^6(h\text{ʎ})^3 (17)$$

For $r = \frac{4}{5}$

$$det(At^2 - Bt - C) =$$

$$-\frac{11175150}{1163046313}t^6 - \frac{19453125}{20040182624}t^5 + \frac{643797}{673875}t^8 - \frac{8704958}{101008985}t^8h^2 + \frac{2013938667}{20201797000}t^7$$
$$+ \frac{898543191}{2885971000}t^8h^3 + \frac{87449193}{459131750}t^7h^2 - \frac{990171}{62738500}t^8h^4 + \frac{1230739029}{23087768000}t^7h^3$$
$$+ \frac{428015825}{27913111512}t^6h + \frac{9523925}{357860404}t^6h^2 - \frac{4453125}{2505022828}t^5h - \frac{6825}{8030528}t^6h^3$$
$$- \frac{703125}{738808576}t^5h^2$$

(18)
Put $h\lambda = H = 0$ in (16),(17),(18)
We have

$$R_1(t,0) = -\frac{5432344}{633555}t^8 + \frac{199656}{23465}t^7 + \frac{1544}{23465}t^6 - \frac{56}{633555}t^5 \quad (19)$$

$$R_2(t,0) = \frac{586800}{254881}t^8 - \frac{2283084}{1274405}t^7 - \frac{2489}{3823215}t^6 + \frac{1}{19116075}t^5 \quad (20)$$

$$R_{\frac{4}{5}}(t,0) = -\frac{11175150}{1163046313}t^6 - \frac{19453125}{20040182624}t^5 + \frac{71533}{74875}t^8 + \frac{2013938667}{20201797000}t^7 \quad (21)$$

Solving the Polynomials (19), (20) & (21) for t. The following table is obtained for the roots of the polynomials.

Table 3.2: Zero Stability of the Proposed Formulae

Step size ratio (r)	**Roots of the proposed methods**
$r = 1$	$t = 0, 0, 0, 0, 0$; -0.0053951501; 0.1777068048; 1
$r = 2$	$t = 0, 0, 0, 0, 0$; -0.0004309211,; 0.7785104297 ; 1
$r = \frac{4}{5}$	$t = 0, 0, 0, 0, 0$; 0.0994658027; 0.1149432390; 1

3.3 A - Stability Region of the Proposed Method

In this section, the region for the absolute stability of the proposed methods is plotted, by considering the stability polynomials (16, 17 & 18). The set of point defined by $t = e^{i\theta}$, $0 \le \theta \le 2\pi$ describes the boundary of the stability region. The following stability region was the complex plot of the proposed methods with the aid of Maple Software.

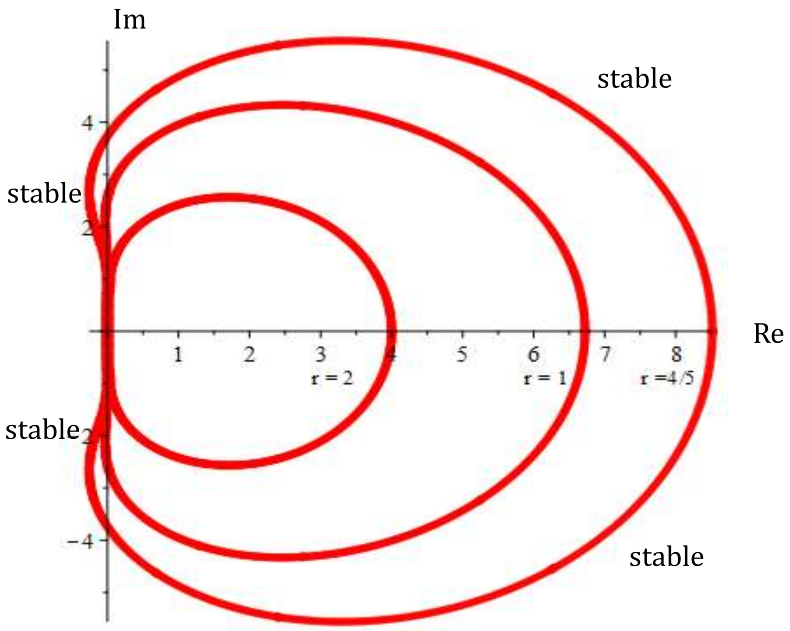

Figure 2: Combine plot for A-stability regions of the proposed methods ($r = 1, 2$ & $r = \frac{4}{5}$)

3.4 Test Problems

To validate the performance of the proposed method(HVSSBBDF), below are some selected stiff IVP of ODEs to consider.

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Table 3.3: Sample of First Order Initial Value Problem of Stiff ODEs

S/n	Problems	Initial Conditions	Interval	Exact Solutions	Eigen Values
1	$y' = -1000(y-1)$	$y(0) = 2$	$0 \le x \le 10$	$y(x) = e^{-1000x} + 1$	-1000
2	$y' = -\frac{y^3}{2}$	$y(0) = 1$	$0 \le x \le 4$	$y(x) = \frac{1}{\sqrt{1+x}}$	
3	$y_1' = 9y_1 + 24y_2 + 5cosx - \frac{1}{3}sinx$ $y_2' = -24y_1 - 51y_2 - 9cosx - \frac{1}{3}sinx$	$y_1(0) = \frac{4}{3}$ $y_2(0) = \frac{2}{3}$	$0 \le x \le 20$	$y_1(x) = 2e^{-3x} - e^{-39x} + \frac{1}{3}cosx$ $y_2(x) = -e^{-3x} + 2e^{-39x} - \frac{1}{3}cosx$	$-3, -39$
4	$y' = 5e^{5x}(y-x)^2 + 1$	$y(0) = 0$	$0 \le x \le 1$	$y(x) = x - e^{-5x}$	
5	$y_1' = -20y_1 - 19y_2$ $y_2' = -19y_1 - 20y_2$	$y_1(0) = 2$ $y_2(0) = 0$	$0 \le x \le 20$	$y_1(x) = e^{-39x} + e^{-x}$ $y_1(x) = e^{-39x} - e^{-x}$	
6	$y_1' = 198y_1 + 199y_2$ $y_2' = -398y_1 - 399y_2$	$y_1(0) = 1$ $y_2(0) = -1$	$0 \le x \le 10$	$y_1(x) = e^{-x}$ $y_2(x) = -e^{-x}$	$-1, -200$

IV. RESULTS AND DISCUSSION

Some chosen problems are solved using the proposed method. The approximate result of the tested problems are put in tables, comparison are made with the existing method to depict the performance of the new scheme. The plotted graphs also highlightedsuperiority of the proposed methods over others considered in this research.The acronyms below are used in the tables.

h= step-size;
MHTD =Method
MAX-ERR = Maximum Error;
EXE-TIME= Execution Time in second;
BBDFO (6) = New Block Backward Differentiation Formula with off-step points of order 6
Ode15s = Variable order Backward Differentiation Formula
3NBBDF = Extended 3 Point Super Class of Block Backward Differentiation Formula
HVSSBBDF = A New Hybrid Variable Step Size Block Backward Differentiation Formula for integrating stiff IVP of ODEs.

Table 4.1: Comparison of Accuracy for Problem 1 & 2

Numerical Result for Problem 1			Numerical Result for Problem 2		
H	**MTHD**	**MAX-ERR**	**H**	**MTHD**	**MAX-ERR**
10^{-3}	BBDFO(6) Ode15s HVSSBBDF	2.11157(−2) 2.08844(−3) 2.31925(−4)	10^{-3}	BBDFO(6) Ode15s HVSSBBDF	5.68483(−7) 9.37878(−4) 3.33469(−4)
10^{-4}	BBDFO(6) Ode15s HVSSBBDF	5.54678(−3) 2.60950(−4) 3.61581(−6)	10^{-4}	BBDFO(6) Ode15s HVSSBBDF	5.71640(−9) 1.14126(−4) 3.34271(−6)
10^{-5}	BBDFO(6) Ode15s HVSSBBDF	7.38966(−5) 3.76862(−5) 4.57182(−8)	10^{-5}	BBDFO(6) Ode15s HVSSBBDF	5.71960(−11) 1.75037(−5) 3.36110(−8)
10^{-6}	BBDFO(6) Ode15s HVSSBBDF	7.60256(−7) 6.32160(−6) 4.32198(−10)	10^{-6}	BBDFO(6) Ode15s HVSSBBDF	9.52614(−11) 2.61573(−6) 3.36256(−10)

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Table 4.2: Comparison of Accuracy for Problem 3 & 4

Numerical Result for Problem 3			Numerical Result for Problem 4		
H	**MTHD**	**MAX-ERR**	**H**	**MTHD**	**MAX-ERR**
10^{-3}	BBDFO(6) Ode15s HVSSBBDF	2.04408(−3) 8.69860(−4) 1.74514(−3)	10^{-3}	3NBBDF RDIBM HVSSBBDF	4.90191(−5) 3.73116(−5) 3.25138(−5)
10^{-4}	BBDFO(6) Ode15s HVSSBBDF	2.28504(−5) 1.21447(−4) 2.13585(−5)	10^{-4}	3NBBDF RDIBM HVSSBBDF	5.20417(−7) 3.73371(−7) 3.25942(−7)
10^{-5}	BBDFO(6) Ode15s HVSSBBDF	2.31054(−7) 1.36101(−5) 2.23564(−7)	10^{-5}	3NBBDF RDIBM HVSSBBDF	5.25030(−9) 3.73652(−9) 3.26109(−9)
10^{-6}	BBDFO(6) Ode15s HVSSBBDF	2.31311(−9) 2.85643(−6) 2.31957(−9)	10^{-6}	3NBBDF RDIBM HVSSBBDF	5.25648(−11) 4.05313(−11) 3.26583(−11)

Table 4.3: Comparison of Accuracy for Problem 5 & 6

Numerical Result for Problem 5			Numerical Result for Problem 6		
H	**MTHD**	**MAX-ERR**	**H**	**MTHD**	**MAX-ERR**
10^{-2}	3NBBDF RDIBM HVSSBBDF	6.98707(−2) 4.45713(−3) 8.43849(−5)	10^{-2}	3NBBDF RDIBM HVSSBBDF	1.94447(−4) 1.52564(−4) 7.13551(−5)
10^{-3}	3NBBDF RDIBM HVSSBBDF	5.40956(−3) 3.74938(−5) 8.45371(−7)	10^{-3}	3NBBDF RDIBM HVSSBBDF	2.07993(−6) 1.76763(−6) 7.32821(−7)
10^{-4}	3NBBDF RDIBM HVSSBBDF	3.08942(−5) 3.52727(−7) 8.47282(−9)	10^{-4}	3NBBDF RDIBM HVSSBBDF	2.09995(−8) 1.79766(−8) 7.59457(−9)
10^{-5}	3NBBDF RDIBM HVSSBBDF	3.18534(−7) 3.31505(−9) 8.50316(−11)	10^{-5}	3NBBDF RDIBM HVSSBBDF	2.10257(−10) 1.82566(−10) 7.60185(−11)
10^{-6}	3NBBDF RDIBM HVSSBBDF	3.19872(−9) 3.11313(−11) 8.52149(−13)	10^{-6}	3NBBDF RDIBM HVSSBBDF	1.41029(−11) 1.85567(−12) 7.89244(−13)

Base on the approximated solutions of the proposed method and other compared schemes presented in table 4.1, 4.2, 4.3 & 4.4 which comprises examples 1, 2, 3 and 4, it was observed that the newly proposed method(HVSSBBDF) outperformed the BBDFO(6) and Ode15s in terms of accuracy in problems 1, 2 and 3. Also, the new scheme has comparative advantage of good accuracy in examples 4, 5 and 6 over 3NBBDF and RDIBM. While, BBDFO(6) has competing advantage over the Matlab solver Ode15s as step size keeps decreasing in example 1. Similarly, BBDFO(6) has clear advantages of good accuracy than Matlab solver Ode15s in example 2 and 3.Likewise, RDIBM is competing with 3NBBDF in terms of accuracy than in example 4 and 6. But, in example 5 RDIBM has favourable accuracy compared to 3NBBDF. Hence, the proposed new scheme (RDIBM) can be an alternative stiff ODEs solver.

V. CONCLUSION

A new hybrid variable step size of block backward differentiation formula for integrating stiff initial value problem of ordinary differential equation was presented. The stability criteria of the proposed method has been investigated, the proposed method is found to be Zero and A Stable, capable of providing two approximate solution values and two off-step points at a time per integrating step. Solutions of some selected stiff IVPS of ordinary differential equations are presented in tabular form and it depicted clearly, that the new scheme has comparative advantages in almost all the problems considered in this research, in terms of accuracy of error. Therefore, thenew method can be a very good solver of first order system of stiff IVP of ordinary differential equations.

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