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Numerical Approach to Picard- Lipschitz Continuity theorem on time scale

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ABSTRACT: This article applied the theory of time scalesintroduced by Stefan Hilger in his PhD thesis to estimat and analysed the forward difference approach and also analyzing the operator ∇Δ, which takes an estimate of improvement leading, to Picard and Lipschitz continuity. The Picard theorem specifically illustrate about a dynamic equation with an initial value resulting to a continuous function with existence and uniqueness of the process that leads to Lipschitz continuity theorem. We also illustrate the process with numerical approach and Picard-Lipschitz theorem was used to illustrate continuous and discrete process with an estimate of approximation. This research reveals a closer relationship betweendynamic equation and ordinary differential equations

Keyword: Picard Lipschitz existence and uniqeness dynamcic equation

I.INTRODUCTION

The theory of time scale was introduced byStefan Hilger in his PhD thesis[19] in order to unify continuous and discrete analysis. Time-scale calculus is a unifies of the theory of difference equations with that of differential equations. It also unifies integral and differential calculus with the calculus of finite differences and this offers a formalism for studying hybrid systems. It has some applications in any field that requires simultaneous modelling of discrete and continuous data. This research is an introduction to the study of the principle by Stefan Hilger analysis. Some important results concerning Picard and Lipschitz theorems, were used to illustrate continuous and discrete process which will serve as problem solving over a long period of time [6, 17].

The general idea of Stefan Hilger is used to prove result for a dynamic equation by concentrating on the concept of time scale. The definition of forward and backward jump operators, △ derivative and △ integration on time scale, will be used in illustrating these definition by examples and we showed how these derivatives and integration coincide with ordinary derivative if $\mathbb{T} = \mathbb{R}$, and the difference derivative and summation if $\mathbb{T} = \mathbb{Z}$. The n^{th} order dynamical system on time scale can be express as follows.

$$f\left(t, x_0^{\Delta^1}, x_0^{\Delta^2}, \ldots x_0^{\Delta^n}, x_1^{\Delta^1}, x_1^{\Delta^2}, \ldots . x_1^{\Delta^n}, x_2^{\Delta^1}, x_2^{\Delta^2}, \ldots x_2^{\Delta^n} \ldots \right) = 0 \quad (1)$$

Where $t \in \mathbb{T}$and $x_1, x_2, x_3 \ . \ . \ .$ n are dependent variable and $X_i: \mathbb{T} \longrightarrow \mathbb{R}$ for $i = 0, 1, 2, 3 \ . \ . \ . n$

This equation produces first, second and high order dynamic equations, capable of solving first, second and higher order dynamic system for continuous and discrete equation respectively. We will concentrate on the dynamic equation on time scale $\mathbb{T}= \mathbb{R}$ and $\mathbb{T}= \mathbb{Z}$,The numerical method to a forward difference approach which take an estimate of improvement and application to the Picard and Lipschitz continuity

1. Differentiation of Time Scales

Take a function:f: $\mathbb{T} \longrightarrow \mathbb{R}$
(where $\mathbb{R}$ could be any Banach space, but is set to the real line for simplicity).

Definition: The delta derivative (also Hilger derivative)$f^{\Delta}(t)$exists if and only if:
For every $\varepsilon > 0$there exists a neighborhood U of tsuch that:

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)\sigma(t) - s| \leq \varepsilon|\sigma(t) - s| \quad (2)$$

for all s in U.

If $\mathbb{T}$ =$\mathbb{R}$, then and $f^{\Delta} = f'$; is the derivative used in standard calculus. If $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t+1, \ \mu(t) = 1, \ f^{\Delta} = \Delta f$ (is the forward difference operator used in difference equations.

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2. IntegrationonTimeScales (Preliminaries)

Definition 1: If a function f: 𝕋 ⟶ ℝ is regulated then the right side limit exist (finite) to all right dense point in 𝕋, Martin Bohner and Allan Peterson [1, 2].

$$X_{rt\rightarrow n} = X^{d}_{rd}(\mathbb{T}) = X^{d}_{rd}(\mathbb{T},\mathbb{R}) \quad (3)$$

Definition2: If f : 𝕋 ⟶ ℝ is a function that is continuous and exist (finite) in the left side limit to all left dense point in 𝕋.

$$X_{Lf\rightarrow n} = X^{d}_{Lf}(\mathbb{T}) = X^{d}_{Ld}(\mathbb{T},\mathbb{R}) \quad (4)$$

Definition 3: Let f: 𝕋 ⟶ ℝ be a right continuous function provided it is continuous to all right dense point in 𝕋,

$$Y_{rd} = Y^{d}_{rd}(\mathbb{T}) = Y^{d}_{rd}(\mathbb{T},\mathbb{R}) \quad (5)$$

Definition 4:A function f: 𝕋 ⟶ ℝ is continuous iff the left side limit exist (finite) at the left dense point in 𝕋.

$$Y_{Lf\rightarrow n} = Y^{d}_{Lf}(\mathbb{T}) = Y^{d}_{Ld}(\mathbb{T},\mathbb{R}) \quad (6)$$

3. HilgerΔDerivative on Time Scale

Definition 3.1 Let f: 𝕋 ⟶ ℝ is continuous at point t ϵ ℝ such thatf^{Δ} (t) be a derivative. Then there exist a neighborhood of a point t – δ and t + δ such that the union

U = {t–δ, t + δ} ⋂ 𝕋 for all δ > 0,

$$|f(\sigma(t) - f(h) - f^{\Delta}(t)\sigma(t) - h)| \leq \epsilon \; |\sigma(t) - h| \quad (7)$$

For all h ϵ U

Theorem 3.1

If f is continuous function on a closed interval [a, b] and differentiable on the open interval (a, b) then if f (a) = f (b), if and only if there exist at least one point c within the interval (a, b) where the

$$f^{\Delta}(c)\, f^{\Delta}(\sigma(c)) \leq 0 \quad (8)$$

Proof

We assume that f^{Δ}(t) is not identically zero since in this case the result is immediate, suppose that f^{Δ}(t) > 0 for some value between a and b, then it follows since f^{Δ}(t) is continuous that it attains its maximum value some were between a and b

Say t, we consider. See Bohner and Peterson[1]

$$f^{\Delta}(t) = \frac{f(\sigma(h+t)) - f(t))}{\sigma(h+t) - t} \leq 0 \quad (9)$$

$$= \lim_{h\rightarrow 0} \frac{f(\sigma(h+t)) - f(t))}{\sigma(h+t) - t} \leq 0$$

For h > 0 then taking the limit h+ t = t through positive value of h, we have f^{Δ} (t), f^{Δ} (σ (t)) ≤ 0.

Suppose if f^{Δ}(t) < 0 so small that t+ h is continuous between a and b, since f^{Δ}(t) is a maximum value it follows that

$$f^{\Delta}(t) = \frac{f(\sigma(h+t)) - f(t))}{\sigma(h+t) - t} \geq 0$$

$$= lim_{h\rightarrow 0} \frac{f(\sigma(h+t)) - f(t))}{\sigma(h+t) - t} \geq 0$$

While if the limit is taken through negative values of h we have

f^{Δ} (t), f^{Δ} (σ (t)) ≥ 0

We referred by definingf^{Δ}(t) as a differentiable function or Hilger delta derivative on time scale 𝕋, provided there exists, t ϵ 𝕋 the functionf^{Δ}: $\mathbb{T}^{k}$ ⟶ ℝ is known as delta derivativeon $\mathbb{T}^{k}$. If f : 𝕋 ⟶ ℝ and f : 𝕋 ⟶ ℤ be a continuous and discrete function over real numbers and integer both satisfy the delta derivative of x^{Δ}at point t ϵ 𝕋. then the delta derivative $x' = x^{\Delta}$which is equal to Δx such that ε > 0.

The definition below satisfies the Hilger Δ derivative on time scale

$$f^{\Delta}(t) = lim_{h\rightarrow t} \frac{f(\sigma(t) - f(t))}{\sigma(t) - t} \leq 0$$

and

$$f^{\Delta}(t) = lim_{h\rightarrow t} \frac{f(\sigma(t) - f(t))}{\sigma(t) - t} \geq 0$$

If 𝕋=ℝ then (t) = t ∀ t ϵ𝕋 hence the above definition is reduces to

$$f^{\Delta}(t) = lim_{h\rightarrow t} \frac{f(h) - f(t)}{h - t} = f'(t)$$

If 𝕋 = ℤ, then f (t + h) = t, so

$$f^{\Delta}(t) = lim_{h\rightarrow t} \frac{f(t+h) - f(t)}{h+t - t} = f(t+h) - f(t) = \Delta f(t)$$

The numerical behavior of Delta has been used to illustrate a right continuous function with a change in the input system; this process which will involve the computing of discrete set of Δf_treferred toΔy_{κ} value of argument using the forward difference approach to determine the argument leading to Picard, Lipschitz continuity theorem such as

$$\Delta y_k = y_{\kappa+1} - y_{\kappa}$$

where there is a change in the input

$$|y_{\kappa+1} - y_{\kappa}|$$

which corresponds to the change in the output

$$|x_{\kappa+1} - x_{\kappa}|$$

with an absolute change in the constant

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Furthermore, the absolute value of the change in input is proportional to the absolute value of the change in output with the proportionality constant equal to the slope M. Reducing the change and make it small in input then the output will also be smaller and M will reduce and satisfies the proportionality in the input and output, which satisfies the Lipschitz condition for continuity [11, 13, 14, 15, 16.18].

Theorem 3.2
A triangle is said to be close bounded subset if and only if the plane is compact, since $f\Delta y_\kappa$ and Δy_κ are continuous and there exists a constant M such that:

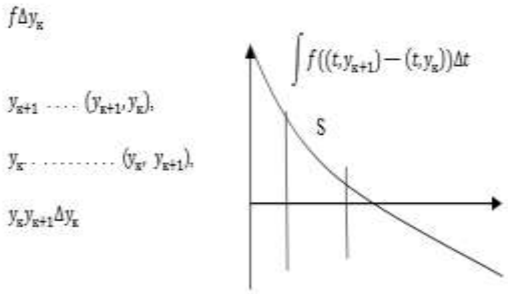

Figure 1.1 Lipschitz continuity theorems

$$|f(y_{\kappa+1}) - f(y_\kappa)| \leq S|y_{\kappa+1} - y_\kappa| \leq S$$

$$|f(y_{\kappa+1} - y_\kappa)| \leq S|y_{\kappa+1} - y_\kappa| \leq S$$

$$= \leq S|y_{\kappa+1} - y_\kappa| = \leq S|\Delta y_\kappa|$$

As illustrated in figure 1

Theorem 3.3 Consider the dynamic equation

$$\left.\begin{array}{l} y^{\Delta}_{\kappa+1} = f(t, y_{\kappa+1}) \\ y_{\kappa+1}(x_\kappa) = y_\kappa k = 0,1,2,3 \,.....\, n \end{array}\right\} \quad y_{\kappa+1}(x) = y_{\kappa+1}$$

Suppose that $f(t, y_{\kappa+1})$ and $y^{\Delta}_{\kappa+1}$ are continuous in some region around the point (x_κ, y_κ) then there is a unique solution to the initial value problem. The function $f(t, y_{\kappa+1})$ is continuous in time t, and defined on an open interval of (x, x_κ). The process is also considered and analysed by Adams Bashford method [7, 9,10].

Proof

$$\int_{x_\kappa}^{x} \frac{y_{\kappa+1}}{\Delta t}\Delta t = \int_{x_\kappa}^{x} f(t, y_{\kappa+1})\Delta t$$

$$y_{\kappa+1}\big|_{x_\kappa}^{x} = \int_{x_\kappa}^{x} f(t, y_{\kappa+1})\,\Delta t \qquad (10)$$

The Picard iteration $y_{\kappa+1}$take one estimate of a solution to a better estimate of solution and for the k^{th} iteration, the desired solution will satisfy$y_{\kappa+1}(x_\kappa) = y_\kappa$which is a fixed point to the Lipschitz condition, helpsto prove convergence and uniqueness of a fixed point in a complete space [3 ,4, 5]].

$$y_{\kappa+1}(x) - y_{\kappa+1}(x_\kappa) = \int_{x_\kappa}^{x} f(t, y_{\kappa+1})\,\Delta t$$

$$y_{\kappa+1} - y_\kappa = \int_{x_\kappa}^{x} f(t, y_{\kappa+1})\,\Delta t \qquad (11)$$

Furthermore if $y_{\kappa+1}$ and y_κ are said to be continuous at the interval $(y_{\kappa+1}, y_\kappa)$, then the equation will eventually lead to Picard theorem that is continuous at each point of the series by substituting k = 0, 1, 2, 3, 4.....n

$$y_{\kappa+1} = y_\kappa + \int_{x_\kappa}^{x} f(t, y_{\kappa+1})\,\Delta t$$

The function which resulted to Picard theorem will lead to the process of Lipschitz continuous theorem.

$$y_{\kappa+1} - y_\kappa = \int_{x_\kappa}^{x} f(t, y_{\kappa+1})\,\Delta t \qquad (12)$$

The function $f(t, y_{\kappa+1})$ is used to illustrate the continuous process in time t, whereby the function will result to the Picard theorem by integration, and then to Lipschitz continuous theorem, with an absolute value on both side of the equation.

$$|y_{\kappa+1} - y_\kappa| = \left|\int_{x_\kappa}^{x} f(t, y_{\kappa+1})\,\Delta t - \int_{x_\kappa}^{x} f(t, y_\kappa)\,\Delta t\right|$$

$$|y_{\kappa+1} - y_\kappa| \leq \int_{x_\kappa}^{x} S|f(t, y_{\kappa+1})\Delta t - f(t, y_\kappa)\Delta t| \qquad (13)$$

A function $(t,\ y_{\kappa+1})$ is said to satisfy Lipschitz condition in the second variable if there exist a constant M > 0 such that

$$|y_{\kappa+1} - y_\kappa| \ \leq \int_{x_\kappa}^{x} S|f(y_{\kappa+1} - y_\kappa)|$$

$$|y_{\kappa+1} - y_\kappa| \leq S|y_{\kappa+1} - y_\kappa| = \leq S|\Delta y_\kappa| \quad (14)$$

Theorem 3:4 A dynamic equation of the form

$$y_{\kappa+1}^{\nabla\Delta} = f(t, y_{\kappa+1}) \qquad \left\{ \; y_{\kappa+1}^{2}(x_\kappa) = y_\kappa^{2} \right.$$

If $f: \mathbb{R}^2 \to \mathbb{R}^2$is bounded above if $y_{\kappa+1}^{\nabla\Delta}$ is bounded provided that $|y_{\kappa+1} - y_\kappa|^2$ is bounded and continuous with local Lipschitz theorem, a numerical interpretation is introduced to analyze the operator $\nabla\Delta$.

Proof

$$y_{\kappa+1}^{\nabla\Delta} = y_{\kappa+1}^{\Delta+1} = y_{\kappa+1}^{\Delta} \cdot y_{\kappa+1}$$

$$= y_{\kappa+1} \int_{x_\kappa}^{x} \frac{y_{\kappa+1}}{\Delta t} \Delta t = y_{\kappa+1}^{2} |_{x_\kappa}^{x} = \int_{x_\kappa}^{x} f(t, y_{\kappa+1}^{2}) \Delta t$$

$$= y_{\kappa+1}^{2}(x) - y_{\kappa+1}^{2}(x_\kappa) = \int_{x_\kappa}^{x} f(t, y_{\kappa+1}^{2}) \Delta t$$

$$|y_{\kappa+1}^{2} - y_\kappa^{2}| = \left| \int_{x_\kappa}^{x} f(t, y_{\kappa+1}^{2}) \Delta t - \int_{x_\kappa}^{x} f(t, y_\kappa^{2}) \Delta t \right| \quad (15)$$

The function $(y_{\kappa+1}^{2}, y_\kappa^{2})$ is said to be bounded above and continuous, with a local Lipschitz theorem if and only if

$$|y_{\kappa+1} - y_\kappa|^2 = \left| \int_{x_\kappa}^{x} f(t, y_{\kappa+1}) \Delta t - \int_{x_\kappa}^{x} f(t, y_\kappa) \Delta t \right|^2$$

$$|y_{\kappa+1} - y_\kappa|^2 \leq \int_{x_\kappa}^{x} S|f(t, y_{\kappa+1})\Delta t - f(t, y_\kappa)\Delta t|^2 \Delta t$$

$$|y_{\kappa+1} - y_\kappa|^2 \leq \int_{x_\kappa}^{x} Sf|y_{\kappa+1} - y_\kappa|^2 \Delta t \to n$$

$$|y_{\kappa+1} - y_\kappa|^2 \leq S \int_{x_\kappa}^{x} |y_{\kappa+1} - y_\kappa|^2 \Delta t \to n \quad (16)$$

Example 3.1

Suppose a non negative continuous function $Y_{K+1}: \mathbb{N}^2 \longrightarrow \mathbb{N}^2$satisfies

$$y_{\kappa+1}^{2} = y_\kappa^{2} + \int_{x_\kappa}^{x} f(t, y_{\kappa+1}^{2}) \Delta t$$

Where

$$y_{\kappa+1}^{2} = y_\kappa^{2} e^{ST}$$

Since the nonnegative function bounded from above by a (constant time); the area covered so far will have exponential growth.

Proof

Then by applying the Bellman Gromwell inequality to determined the existence and uniqueness that is bounded above with a local Lipschitz theorem.

$$y_{\kappa+1}^{2} = y_\kappa^{2} + \int_{x_\kappa}^{x} f(t, y_{\kappa+1}^{2}) \Delta t$$

$$x_{\kappa+1}^{2} = x_\kappa^{2} + \int_{x_\kappa}^{x} f(t, x_{\kappa+1}^{2}) \Delta t$$

By subtracting both Picard equations $y_{\kappa+1}^{2}$and $x_{\kappa+1}^{2}$, the following procedure will lead to Lipschitz continuity process

$$y_{\kappa+1}^{2} - x_{\kappa+1}^{2} = y_\kappa^{2} - x_\kappa^{2} + \int_{x_\kappa}^{x} f(t, y_{\kappa+1}^{2}) \Delta t - \int_{x_\kappa}^{x} f(t, x_{\kappa+1}^{2}) \Delta t$$

$$|y_{\kappa+1} - x_{\kappa+1}|^2 = |y_\kappa - x_\kappa|^2 + \left| \int_{x_\kappa}^{x} f(t, y_{\kappa+1}) \Delta t - \int_{x_\kappa}^{x} f(t, x_{\kappa+1}) \Delta t \right|^2 \quad (17)$$

The above equation is a Lipschitz continuity theorem and by illustrating Gromwell's inequality, which is bounded and continuous to a fixed point.

$$|y_{\kappa+1} - x_{\kappa+1}|^2 = |y_\kappa - x_\kappa|^2 + \int_{x_\kappa}^{x} S|f(t, y_{\kappa+1})\Delta t - f(t, x_{\kappa+1})\Delta t|^2$$

$$|y_{\kappa+1} - x_{\kappa+1}|^2 = |y_\kappa - x_\kappa|^2 + \int_{x_\kappa}^{x} S|y_{\kappa+1} - x_{\kappa+1}|^2 \Delta t$$

$$|y_{\kappa+1} - x_{\kappa+1}|^2 \leq |y_\kappa - x_\kappa|^2 e^{S(x - x_\kappa)} \quad (18)$$

If $|y_\kappa - x_\kappa|^2 = 0.1^2$ where $|y_\kappa - x_\kappa| = 0.01$such that$S = 0$, since the solution is bounded above and the interval guarantee the existence of locally Lipschitz neighborhood, then

$$|y_{\kappa+1} - x_{\kappa+1}|^2 \leq |y_\kappa - x_\kappa|^2 e^{S(x - x_\kappa)}$$

Similarly,

$$|y_{\kappa+1} - x_{\kappa+1}|^2 \leq |y_\kappa - x_\kappa| e^0$$

Such that

$$|y_{\kappa+1} - x_{\kappa+1}|^2 \leq 0.01 e^0$$

$$|y_{\kappa+1} - x_{\kappa+1}|^2 \leq 0.01 \quad (19)$$

The solution is said to be bounded above, between an interval $(y_{\kappa+1}, x_{\kappa+1})$ and (y_κ, x_κ) with a locally Lipschitz that grow gradually to a fixed point of 0.01 between the interval $|y_{\kappa+1} - x_{\kappa+1}|$that give guarantee to the stability of the system with a global existence and uniqueness of the process.

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II. CONCLUSION

The research considered numerical approach to Picard-Lipschitz continuity theorem on time scale. The Picard specifically illustrate about a dynamic equation with an initial value resulting to a continuous function with existence and uniqueness of the process that leads to Lipschitz continuity [8,12]. We also illustrate the process with numerical approach and Picard- Lipschitz continuity theoremwas used to illustrate continuous and discrete process with an estimate of approximation. This research reveals a closer relationship between the dynamic equation to that of the ordinary differential equations.

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