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Contribution of Vector Calculus in Electrical Engineering

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ABSTRACT:This paper shows how Mathematics and Physics through vector calculus are very useful in the success of engineering especially electrical technology. Using different vector operators, some electrical engineering problems are solved. These vector operators named gradient, divergence; Curl and Laplacian operator have been elaborated from different authors' literatures with their corresponding applications in Electrical Technology. The vector calculus in electrical technology is mostly applied to describe Maxwell's equations named Gauss's law for electric flux, Gauss's law for magnetism, Faraday's law, and Maxwell-Ampere's law which are essential tools of electrical engineers to design electrical and electronics equipment that are used to make for example cell phones, satellites, televisions and computers. In this paper, Maxwell's equations cover most useful applications of vector calculus.

Keywords: Vectors, operators, Electrical technology, Maxwell's equations.

I. INTRODUCTION

In engineering, mathematics and physics are more useful in daily life. Those who are in engineering fields need to be strong in mathematics and Physics so that they perform well their engineering tasks.

The students need solid knowledge of basic principles, methods, and results, and a clear view of what engineering mathematics is all about, and that it requires proficiency in all three phases of problems solving: (a) Modeling which is to translate into a mathematical model, (b) Solving the model using suitable mathematical method, and (c) Interpreting Mathematical result in physical or other terms to see what it practically means and implies [1]. One among different topics which are useful especially in electrical engineering is Vector Calculus. Vector calculus is used to model a vast range of engineering phenomenon including electrostatic charges, electromagnetic field, Airflow around aircraft, cars and other solid objects, fluid flows around ships and heat flow in nuclear reactors [2]. The author in [3] mentioned that, Fluid mechanics, Maxwell's wave, heat equations and electromagnetic theory need mathematical tool. The analysis of different functions using curl, gradient, divergence and Laplacian are some of mathematical tools to solve electrical engineering problems. The focus of this paper is on above mentioned mathematical tools from vector calculus (vector and scalar fields) and applications in electrical engineering.

II. MATHEMATICAL VECTOR OPERATORS AND APPLICATIONS

The vector operators are applied in electromagnetism, and much of electromagnetism is concerned with solving Maxwell's equations for different boundary conditions as shown in [4, 5, 2]. Maxwell's Equations provide a complete description of electromagnetic phenomena and underpin all modern information and communication technologies. Maxwell's Equations are the essential tool of electrical engineers, used to design all electrical and electronic equipment from cell phones to satellites, televisions to computers and power stations to washing machines [6]. Maxwell's equations are a set of four mathematical equations that relate precisely the electric and magnetic fields to their sources, which are electric charges and currents. Maxwell's equations in differential form describe field behavior at a point in a continuous medium [7]. The authors in [8, 7, 2] commonly describe in differential form four Maxwell's equations: Gauss's law for electric flux, Gauss's law for magnetism, Faraday's law and Ampere's law. In this section, the vectors operators are described with applications in electromagnetism.

1. Partial differentiation of vectors

The partial derivative of a vector $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$ with respect to x is defined by as $\frac{\partial \vec{v}}{\partial x} = \frac{\partial v_x}{\partial x}\vec{i} + \frac{\partial v_y}{\partial x}\vec{j} + \frac{\partial v_z}{\partial x}\vec{k}$, where v_x, v_y and v_z are the differentiable functions of x, y and z. The partial derivatives with respect to y and z is defined in a similar way, as are higher partial derivatives. For example: $\frac{\partial^2 \vec{v}}{\partial x^2} = \frac{\partial^2 v_x}{\partial x^2}\vec{i} + \frac{\partial^2 v_y}{\partial x^2}\vec{j} + \frac{\partial^2 v_z}{\partial x^2}\vec{k}$ and this is a second partial derivative with respect to x [2, 1].

Electrostatic potential

In electrostatics, the electric potential (V) is a scalar and the intensity of the electric field ($\vec{E}$) is a vector. If the electric potential in a region is the same at different points in the region then there is no electric field in that region. If the electric potential is different at different points in the region then the electric field exists in that region. The direction of the electric field is in the direction in which the potential decreases. The relation between the electric potential and the electric field is given by $\vec{E} = -\frac{\partial V}{\partial r}\hat{r}$, where $\hat{r}$ is the unit vector along the direction of field and the negative sign shows that the field is in the direction of the decreasing potential. Thus, the gradient makes the relation between the vector and the scalar physical quantities [9]. Engineers working on the design of equipment such as cathode ray tubes and electrical valves, which are also commonly known by their generic term vacuum tubes, need to calculate the electrostatic potential that results from an accumulation of static charges at various points in a region of space. Complicated examples require the use of a computer [2].

2. Gradient of a scalar field

Given a scalar function ϕ of x, y and z variables. The gradient of ϕ denoted by $\nabla\phi$ or grad ϕ is given by $\nabla\phi = \text{grad } \phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$. It is often useful to write gradient of ϕ in the form $\nabla\phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$ where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is called a vector operator**,** say, Nabla or Del operator and is regarded as operating on the scalar ϕ. The process of forming a gradient applies only to a scalar field and result is always a vector field [10, 11, 12, 2, 1]. The gradient $\nabla\phi$ is in the direction of the steepest increase in ϕ whereas its magnitude is equal to the rate of increase in that direction [11].

3. Divergence of a vector field

Given a vector field $\vec{v} = v(x, y, z)$. If $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$ and differentiating partially with respect to x, y and z, respectively and adding the obtained quantities the result is a scalar quantity called divergence of $\vec{v}$ denoted by $\text{div } v$ or $\nabla\cdot\vec{v}$ and it is given by $\text{div } \vec{v} = \nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$. If we use the vector operator notation and the scalar product (not in usual sense because $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is a vector operator) to evaluate

$$\nabla\cdot\vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).\vec{v}$$
$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(v_x\vec{i} + v_y\vec{j} + v_z\vec{k}\right).$$
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

The process of finding the divergence is always done on a vector field and the result is always a scalar field. [10, 12, 2, 1]. The divergence of a vector field is also a scalar field that represents the flux generation per unit volume at each point of the field. It means that the divergence measures how much a vector field spreads, or diverges, from a point. The value of the divergence of a vector field at a particular point gives a measure of the "net mass flow" or "flux density" of the vector field in or out of that point. To understand what such a statement means, imagine that the vector field $\vec{v}$ represents velocity of a fluid [13].
If $\nabla\cdot\vec{v} = 0$ at a point (x, y, z), the rate at which fluid is flowing into that point (sink) is equal to the rate at which fluid is flowing out (source) [13], then $\vec{v}$ is said to be divergence free, no source or sink [10]. At a point (x, y, z), positive divergence ($\nabla\cdot\vec{v} > 0$) signifies more fluid flowing out than in, so it is a source. At a point (x, y, z), negative divergence ($\nabla\cdot\vec{v} < 0$) signifies more fluid flowing in than out, so is a sink. [10, 13].

Gauss's law for electric flux: $\nabla\cdot\vec{D} = \rho$

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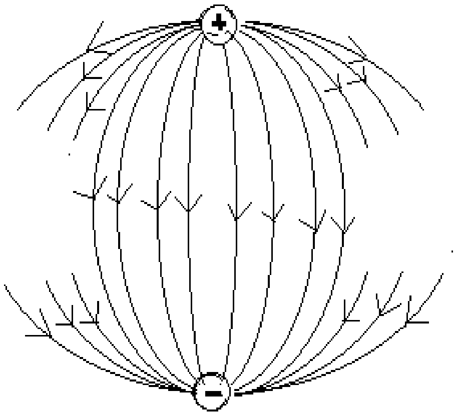
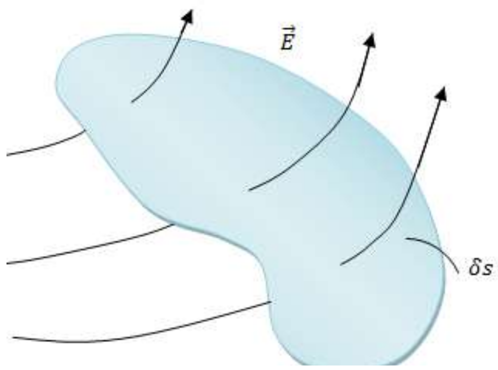
where, $\vec{D}=\varepsilon_0\vec{E}$ is electric flux density with ε_0 the permittivity of free space and ρ is charge density. This Maxwell's equation is a general form of Gauss's theorem which states that the total electric flux flowing out of a closed surface is proportional to the electric charge enclosed by that surface [8, 2]**.** It describes the electric force field surrounding a distribution of electric charge ρ. It shows that the electric field lines diverge from areas of positive electric charge and converge onto areas of negative electric charge [6], [4].



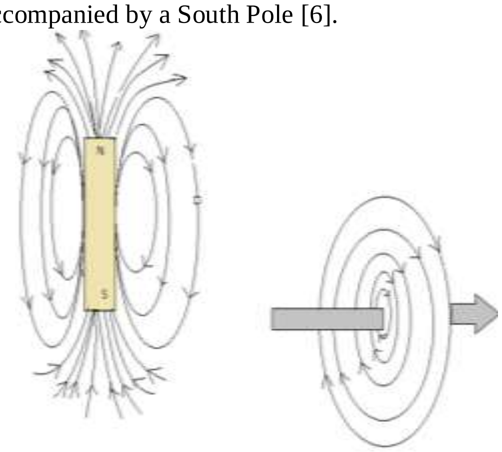
Suppose we surround a region containing charges with a surface S. If a small portion of this surface δS is chosen, we can draw the field lines which pass through this portion as shown in the figure below [2].



The flux of electric field $\vec{E}$ through δS is a measure of the number of lines of force passing through δS. Gauss's law states that the total flux out of any closed surface **S** is proportional to the total charge enclosed.

Gauss's law for magnetism: $\nabla\cdot\vec{B}=0$

where, $\vec{B}$ is the magnetic flux. This Maxwell's equation arises from the observation that all magnetic poles occur in pairs and therefore magnetic field lines are continuous; that is, there are no isolated magnetic poles [2]. It shows that magnetic field lines curl to form closed loops, with the implication that every North Pole of a magnet is accompanied by a South Pole [6].



4. Curl of a vector field

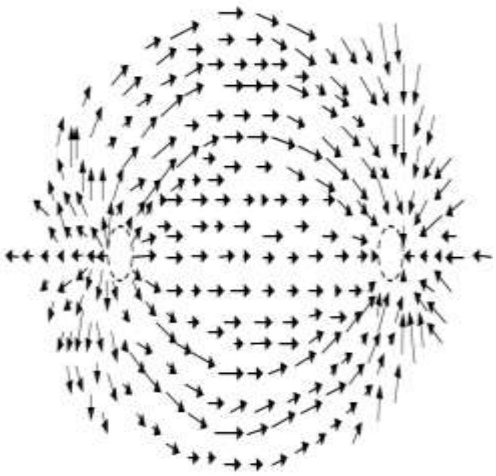
From [12, 2] the Curl of a vector field $\vec{v}$ denoted by $curl\ \vec{v}$ or $\nabla\times\vec{v}$ measures how much a vector field rotates or curls around a point and it is defined by

$$curl\ \vec{v}=\nabla\times\vec{v}=\nabla\times\left(v_x\vec{i}+v_y\vec{j}+v_z\vec{k}\right)$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}$$

$$=\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\vec{i}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\vec{j}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\vec{k}.$$

As mentioned in [2] this determinant is evaluated as usual but $\frac{\partial}{\partial x},\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are operators, not multipliers. Imagine that $\vec{F}$ represents the velocity of a stream or lake. Drop a small twig in the lake and watch its travel. The twig may perhaps be pushed by the current so that it travels in a large circle, but the curl will not detect this. What curl $\vec{F}$ measures is how quickly and in what orientation the twig itself rotates as it moves [13]. If the vector field $\vec{v}$ under consideration represents a fluid flow then it may be shown that curl $\vec{v}$ is a vector which measures the extent to which individual particles of the fluid are spinning or rotating. Then, a vector field $\vec{v}$ whose curl is zero $(\nabla\times\vec{v}=0)$ for all values of x, y and z is said to be irrotational or conservative, [14, 13, 2, 1].For a two dimensional flow with F represents the fluid velocity, $\nabla\times\vec{v}$ is perpendicular to the motion and represents the direction of axis of rotation [14]. The **curl of a vector field** at a particular point in space represents the tendency of the field to swirl around that point. That is the tendency to follow a non-linear direction. These regions correspond to a large curl value and indicate that the field swirls significantly clockwise (negative curl) or anti-clockwise (positive curl) [15].



Faraday's law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

where, $\vec{E}$ is the electric field strength and $\vec{B}$ is the magnetic flux density. This Maxwell's equation describes how electric and magnetic fields are related and how a time-varying magnetic field will cause an electric field to curl around it [6, 2]. The analogue of Faraday's law relates the rate of change of the flux of an electric field through a surface to the circulation of the magnetic field around the perimeter of surface. Maxwell added a correction term due to the flux of electric current because of the moving electric charges through the surface [8]. The mathematical formulation of Faraday's magneto-electric induction is called Faraday's law, which states that the rate of change of a magnetic field through a two dimensional surface is proportional to the circulation of the electric field around a one dimensional perimeter of the two dimensional surface [8].

Ampère law: $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

where, $\vec{J}$ is the free current density, $\vec{D} = \varepsilon_0 \vec{E}$ is electric flux density with $\vec{E}$ the electric field strength and ε_0 the permittivity of free space, $\vec{H} = \frac{\vec{B}}{\mu_0}$ is the magnetic field strength with $\vec{B}$ the magnetic flux and μ_0 a permeability of free space as shown in [16] .

This equation describes how electric and magnetic fields are related and how a magnetic field curls around a time-varying electric field or an electric current flowing in a conductor.[6, 17].The author in [6] mentioned that Maxwell's equations can explain how your hair stands on end when you remove your nylon sweater, how a compass needle always points north, how a power station turbine can generate electricity, and how a loudspeaker can convert an electric current into sound. When combined, these equations also describe the transmission of radio waves and the propagation of light. All magnetic fields circulate back upon themselves, and its curl is non-zero at **exact locations** through which an electric current is flowing or an electric field is changing. The solutions of Maxwell's equations are essential for the analysis, design and advancement of wireless devices and system, high-speed electronics, microwave imaging, remote sensing,…etc [4].As shown in [8] the wave requires both electric and magnetic fields to propagate and they co-propagate. This means that the magnetic field provides the medium for propagation of the electric field and vice versa. Also, there is no limitation on the possible frequencies of the waves, this implies that the allowed wavelengths are in the interval $\infty > \lambda \geq 0$ and the average power of the light wave per unit area is the irradiance and is determined by the Pointing vector.

5. Laplace and Poisson's equations

If two operators, gradient and divergence are combined to denote $\nabla \cdot (\nabla \phi)$ or $\nabla^2 \phi$ pronounced del-squared ϕ or simply denote $\Delta \phi$ and obtain

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$ where,

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

the operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian, [5, 2]. Equating by zero, $\nabla^2 \phi = 0$, equivalently $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$. This obtained equation is Laplace's equation [2]**.** As mentioned in [2], engineers sometimes need to solve complex electrostatic problems when designing electrical equipment. The calculation of the electrical field strength in a high-voltage electrical distribution station to ensure there is no danger of electrical discharge across the air gap between components that have different voltages; Poisson's equation is useful when carrying out this work.

If $\vec{E}$ is an electric field and $\vec{V}$ an electrostatic potential, then the two fields are related by $\vec{E} = -\nabla\vec{V}$ and combining with Gauss's law $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ we get the Poisson's equation $\nabla^2\vec{V} = -\frac{\rho}{\varepsilon_0}$. Solving Poisson's equation, the electrostatic potential in a region occupied by charges is determined. In a charge-free region $\rho = 0$, the Poisson's equation reduces to Laplace's equation $\nabla^2\vec{V} = 0$ [2].

Antenna coordination

To accurately compute the electromagnetic properties of wire antennas, it is necessary to solve Maxwell's equation subject to boundary conditions imposed by the wires and their surrounding environment. In order to simplify the problem, it is usually and customary adequate to assume that the wire antenna is located in free space and to ignore to first order any radiative interactions between one part of the wire array and another. This approximation works well for the most wire antennas less than a wavelength across. Since Maxwell's equations are linear, the radiation from such wire antennas can be approximated as the linear superposition of the radiation produced by each infinitesimal sub element of such antenna. Such sub-element act as short dipole antennas [18].

III. CONCLUSION

In this paper, we more focus on the contribution of vector calculus including vector operators such as gradient, divergence, Curl and Laplacian operator and how they are coordinated with electromagnetism and Maxwell's equations. Since the general aim of our paper is to specify the contribution of vector calculus in Electrical engineering; and because Electrical Engineering requires the knowledge about vector calculus. It means that if vector calculus is left out Electrical Engineering, those who pursue engineering related courses can't claim to have effectively learned the course. Finally, Vector Calculus is important and has a clear intervention in Electrical Engineering through its contributions.

Those who are close to Electrical Engineers are recommended that vector calculus is matter a lot in Electrical Engineering. Electrical Engineers should therefore master vector calculus for them to be successful in their career since vector calculus has a clear and significant contribution to Electrical Engineering.

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