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Application of General Integral Transform of Error Function for Evaluating Improper Integrals

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ABSTRACT:
Generally, error function occurs in Probability, Physical sciences , Engineering ,Statistics and biological sciences . Many researchers use various integral transforms of error functions to evaluate improper integrals. In this article we study General integral transform of error functions. In applications part we use general integral transform of error function to evaluate improper integrals containing error functions.

Keywords: General Integral transform, Error functions, Improper integral containing Error functions.
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I. INTRODUCTION:

Error and Complimentary error functions are found in the solutions of many advanced problems in Engineering, like heat and mass transfer problems, Vibrating beam problems. when we want to solve types of problems by using any integral transform, we must know the integral transform of the error functions. Integral transform of error functions play an important role in evaluating improper integrals containing error function, without any tedious calculations. Till now many researchers have developed several integral transforms Laplace, Kamal, Sadik, Elzaki, Aboodh, Mohand, Rishi, Tarig transforms. Integral transforms are very much useful to solve many advanced problems of science and engineering such as Radioactive decay problems, Heat conduction problems, problems of motion of the particle under gravity ,Vibration of beam, Problems in electric circuits,etc. Recently S.R Kushare and D.P.Patil [1] introduce Kushare transform in September 2021.In October 20121, S.S.Khakale and D.P.Patil [2] introduce Soham transform. As researchers are going introducing new integral transforms at the same time many researchers are interested to apply these transforms to various types of problems. In January 2022,R.S.Sanap and D.P.Patil [3] used Kushare transform to solve the problems based on Newton's law of cooling. In April 2022 D.P.Patil etc. [4] use Kushare transform to solve the problems on growth and decay. In October 2021 D.P.Patil [5] used Sawi transform in Bessel function.D.P.Patil [6] used Sawi transform of error function for evaluating improper integral further,Lpalce and Shenu transforms are used in chemical science by D.P.Patil [7].Dr. Patil [8] solved the wave equation by Sawi transform and its convolution theorem. Further Patil [9] also used Mahgoub transform for solving parabolic boundary value problem.Dr. Patil [10] obtain solution of the wave equation by using double Laplace and double Sumudu transform. Dualities between double integral transforms are derived by D.P.Patil [11].Laplace, Elzaki, and Mahgoub transforms are used for solving system of first order and first degree differential equations by Kushare and Patil [12].Boundary value problems of the system of ordinary differentiable equations are by using Aboodh and Mahgoub transform by Patil [13].D.P.Patil [14] study Laplace, Sumudu, Elzaki and Mahgoub transforms comparatively and apply them in Boundary value problems. Parabolic Boundary value problems are also solved by D. P. Patil [15].For that he used double Mahgoub transform. Soham transform is used to obtain the solution of system of differential equations by D. P. Patil etal [16]. D. P. Patil et al also used Soham transform for solving Volterra integral equations of first kind[17].

This paper is orgainsed as follows. Second section is used for preliminaries. Error functions are discussed in third section. Fourth section is devoted for general integral transforms of error functions. Fifth section contains general integral transforms of complimentary error functions.

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II. PRELIMINARIES:

2.1. A New General Integral Transform: [18]

Definition: Let f(t) be a integral function defined for $t \geq 0$,p(s) $\neq$ 0 and q(s) are positive real functions, we define the new general integral transform T(s) of f(t) by the formula

$$T\{f(t);s\} = T(s) = P(s)\int_0^\infty f(t)e^{-q(s)t}dt \quad \ldots\ldots\ldots\ldots(1)$$

Provided the integral exists for some q(s).

1.2. Some Useful Properties of New General Integral Transform: [18]

(1) Linearity Property:

If T $\{v_1(t)\} = R_1(p(s),(q(s))$ and $T\{v_2(t)\} = R_2(p(s),q(s))$, then T $\{av_1(t)+bv_2(t)\} = aR_1(p(s),q(s))+bR_2(p(s),q(s))$,

Where a, b are arbitrary constants .

(2) Scaling property:

If T {v(t)} =R(p(s),q(s)) , then T $\{v(at)\} = \frac{1}{a}R(p(s), \frac{q(s)}{a})$ with $a \neq 0$.

(3) First shifting property :

If T { v(t)} = R(p(s),q(s)), then T $\{e^{at}$ v(t) } = R(p(s),q(s)-a).

(4) Second Shifting Property :

If T {v(t)} = R(p(s),q(s)) ,then T {v(t-a)H(t-a)}= $e^{-aq(s)}$ R(p(s),q(s)).

(5) New General Integral Transform of function t F(t)

Let p(s) & q (s) are differentiable and q' (s) $\neq$ 0 , then {t F(t) } = $-\frac{p(s)}{q'(s)}\left(\frac{T(s)}{p(s)}\right)'$

(6) Convolution Theorem for New General Integral Transform:

Let $f_1(t)$ and $f_2(t)$ have new integral transform F_1 (s) and F_2 (s) .Then the new integral transform

of the Convolution of f1 and f2 is

$$f_1 * f_2 = \int_0^\infty f_1(t)f_2(t-\tau)d\tau = \frac{1}{p(s)}F_1(s).F_2(s)$$

(7) Let p(s) and q(s) are differentiable (q'(s) $\neq$ 0), and f(t) ϵ c^n, then

$$T\{t f^{(n)}(t)\} = -\frac{p(s)}{q'(s)}\frac{d}{ds}\left[\frac{1}{p(s)}T\{t f^{(n)}(t)\}\right]$$

1.3. New General Integral Transform of some function:[18]

Sr. No	Function $f(t)=T^{-1}\{T(s)\}$	New Integral Transform $T(s)=T\{f(t);s\}$
1	1	$\frac{p(s)}{q(s)}$
2	T	$\frac{p(s)}{q(s)^2}$
3	t^α	$\frac{\Gamma[\alpha+1]p(s)}{q(s)^{\alpha+1}}$,α>0
4	Sint	$\frac{p(s)}{q(s)^2+1}$
5	Cost	$\frac{q(s)p(s)}{q(s)^2+1}$
6	Sinhkt	$\frac{kp(s)}{[q(s)]^2-k^2}$
7	Coshkt	$\frac{p(s)q(s)}{[q(s)]^2-k^2}$
8	e^{at}	$\frac{p(s)}{q(s)-a}$, q(s)>a
9	tH(t-1)	$\frac{e^{-q(s)}(q(s)+1)p(s)}{q(s)^2}$
10	f'(t)	q(s) T (s)-p(s)f(0)

III. ERROR AND COMPLEMENTARY ERROR FUNCTION :

We define error and complementary Error function as follow

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

And

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erf c(x)= $\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}$ dt ………………...(3)

Important Properties of Error and Complementary Error function:

i) The Sum of error and Complementary error functions is unity : erf (x) + erf c (x)=1.

ii) Error function is an odd function: erf(-x) = - erf x.

iii) The value of error function at x = 0 is 0.

iv) The Value of complementary error function at x = 0 is 1.

v) The value of error function. at x = ∞ is 1

vi) The value of Complementary error function at x =∞ is 0.

IV. NEW GENERAL INTEGRAL TRANSFORM OF ERROR FUNCTION:

By putting x= $\sqrt{t}$ *in* equation (2), we have erf$\sqrt{t}$ = $\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}$ dx = $\frac{2}{\sqrt{\pi}}\int_0^t\{1-\frac{x^2}{1!}+\frac{x^4}{2!}-x63!+\ldots\}dx$

$$=\frac{2}{\sqrt{\pi}}[x-\frac{x^3}{1!}+\frac{x^5}{2!5}-\frac{x^7}{3!7}+\cdots]$$

$$\therefore erf\sqrt{t}=\frac{2}{\sqrt{\pi}}[\sqrt{t}-\frac{t^{3/2}}{1!3}+\frac{x^{5/2}}{2!5}-\frac{x^{\frac{7}{2}}}{3!7}+\cdots]$$ …………………(4)

applying new general integral transform to both sides of equation (4), we get

$$T\{erf\sqrt{t}\}=\frac{2}{\sqrt{\pi}}T[\sqrt{t}-\frac{t^{3/2}}{1!5}+\frac{x^{5/2}}{2!7}-\frac{x^{\frac{7}{2}}}{3!7}+\cdots]$$ …………………(5)

Applying linearity property of new integral transform on equation (5), we

$$T\{erf\sqrt{t}\}=\frac{2}{\sqrt{\pi}}\left[\frac{\Gamma[\frac{1}{2}+1]p(s)}{q(s)^{\frac{1}{2}+1}}-\frac{\Gamma[\frac{3}{2}+1]p(s)}{q(s)^{\frac{3}{2}+1}}+\frac{\Gamma[\frac{5}{2}+1]p(s)}{q(s)^{\frac{5}{2}+1}}-\frac{\Gamma[\frac{7}{2}+1]p(s)}{q(s)^{\frac{7}{2}+1}}+\cdots\right]$$

$$s=\frac{2}{\sqrt{\pi}}\left[\frac{\Gamma\frac{3}{2}p(s)}{q(s)^{\frac{3}{2}}}-\frac{\Gamma\frac{5}{2}p(s)}{q(s)^{\frac{5}{2}}}+\frac{\Gamma\frac{7}{2}p(s)}{q(s)^{\frac{7}{2}}}-\frac{\Gamma\frac{9}{2}p(s)}{q(s)^{\frac{9}{2}}}+\cdots..\right]$$

$$=\frac{2}{\sqrt{\pi}}\left[\frac{\frac{1}{2}\sqrt{\pi}\,p(s)}{q(s)^{\frac{3}{2}}}-\frac{\frac{3}{4}\sqrt{\pi}\,p(s)}{1!3q(s)^{\frac{5}{2}}}+\frac{\frac{15}{8}\sqrt{\pi}\,p(s)}{2!5q(s)^{\frac{7}{2}}}-\frac{\frac{105}{16}\sqrt{\pi}\,p(s)}{3!7q(s)^{\frac{9}{2}}}+\cdots\right]$$

$$=\frac{2}{\sqrt{\pi}}\ x\frac{\sqrt{\pi}}{2}\left[\frac{p(s)}{q(s)^{\frac{3}{2}}}-\frac{\frac{3}{2}p(s)}{3q(s)^{\frac{5}{2}}}+\frac{\frac{15}{4}p(s)}{2!5q(s)^{\frac{7}{2}}}-\frac{\frac{105}{8}p(s)}{3!7q(s)^{\frac{9}{2}}}+\cdots..\right]$$

$$=\frac{p(s)}{q(s)^{\frac{3}{2}}}[1-\frac{1}{2q(s)}+\frac{3}{8q(s)^2}-\frac{5}{16q(s)^3}+\cdots]$$

$$=\frac{p(s)}{q(s)^{\frac{3}{2}}}[1+\frac{1}{q(s)}]^{-\frac{1}{2}}=\frac{p(s)}{q(s)^{\frac{3}{2}}}[\frac{q(s)+1}{q(s)}]^{-\frac{1}{2}}=\frac{p(s)}{q(s)^{\frac{3}{2}}}\frac{\sqrt{q(s)}}{\sqrt{q(s)+1}}$$

$$\therefore T\{erf\sqrt{t}\}=\frac{p(s)}{q(s)\sqrt{q(s)+1}}$$ ………………(6)

V. NEW GENERAL INTEGRAL TRANSFORM OF COMPLEMENTARY ERROR FUNCTION :

We have, erf c $\sqrt{t}$ = 1-erf $\sqrt{t}$ ……..(7)

Applying new general integral transform both sides. of equation (7) we get,

T {erf c $\sqrt{t}$}= T {1- erf $\sqrt{t}$} .....(8)

Applying the linearity Property of new general integral transform on equation (8), we get,

T {erf c $\sqrt{t}$}= T {1}- T {erf $\sqrt{t}$} $=\frac{p(s)}{q(s)}-\frac{p(s)}{q(s)\sqrt{q(s)+1}}$

VI. APPLICATIONS:

6..1 Evaluate the improper integral I=$\int_0^\infty e^{-t}$ *erf* $\sqrt{t}$ *dt*

Solution: We have, T [erf $\sqrt{t}$] =$\frac{p(s)}{q(s)\sqrt{q(s)+1}}$

Taking q(s) = 1 and applying new general integral transform, we get

p(s)$\int_0^\infty e^{-t}\ erf\sqrt{t}\,dt=\frac{p(s)}{1\sqrt{1+1}}=\frac{p(s)}{\sqrt{2}}$

$$\therefore\int_0^\infty e^{-t}\,erf\sqrt{t}\,dt=\frac{1}{\sqrt{2}}$$

6.2 Evaluate the improper integral I =$\int_0^\infty te^{-3t}$ erf $\sqrt{t}$ dt

Solution: We have, T [erf $\sqrt{t}$] =$\frac{p(s)}{q(s)\sqrt{q(s)+1}}$

T {t.erf $\sqrt{t}$} = $\frac{-p(s)}{q'(s)}$ [$\frac{p(s)}{q(s)\sqrt{q(s)+1}}$ x $\frac{1}{p(s)}$] = $\frac{-p(s)}{q'(s)}$ [$\frac{1}{q(s)\sqrt{q(s)+1}}$]

T{t.erf $\sqrt{t}$} = -p(s)[$\frac{-1}{q(s)^2\sqrt{q(s)+1}}-\frac{1}{2q(s)(q(s)+1)^{\frac{3}{2}}}$] ……….(9)

By applying definition of new general integral transform we get,

T {t.erf $\sqrt{t}$}= p(s)$\int_0^\infty t\ erf\sqrt{t}\ e^{-q(s)t}$ dt ....(10)

From equation (9) & (10) We get,

p(s)$\int_0^\infty t\ erf\sqrt{t}\ e^{-q(s)t}$ dt = -p(s)[$\frac{-1}{q(s)^2\sqrt{(q(s)+1)}}-\frac{1}{2q(s)(q(s)+1)^{\frac{3}{2}}}$]

Taking q(s)=3 in above equation

p(s)$\int_0^\infty t\, erf\,\sqrt{t}\; e^{-3t}$ dt = -p(s)[$\frac{-1}{9\sqrt{4}} - \frac{1}{6(4)^{\frac{3}{2}}}$] =-p(s)[-$\frac{-66}{864}$] =-p(s) $\left[\frac{11}{144}\right]$

$$\therefore \int_0^\infty t\, erf\,\sqrt{t}\; e^{-3t}\, dt = \frac{11}{144}$$

6.3 Evaluate the Improper Integral I = $\int_0^\infty e^{-5t}$[erf $\sqrt{t}$ * erf $\sqrt{t}$]dt

Solution: We know that, T {erf $\sqrt{t}$} =$\frac{p(s)}{q(s)\sqrt{q(s)+1}}$

By the convolution theorem of new general integral transform we have

T {erf $\sqrt{t}$* erf $\sqrt{t}$}=$\frac{1}{p(s)}$ $x\,\frac{p(s)}{q(s)\sqrt{q(s)+1}}\,x\,\frac{p(s)}{q(s)\sqrt{q(s)+1}}$

=$\frac{p(s)}{q(s)^2(q(s)+1)}$ = $\frac{p(s)}{q(s)^3 q(s)^2}$ ….(11)

By the definition of new general integral transform , we have,

T {erf $\sqrt{t}$ * ert $\sqrt{t}$}= p(s) $\int_0^\infty e^{-q(s)t}$ [erf $\sqrt{t}$* erf $\sqrt{t}$] dt ....(12)

from equation 11 & 12 , we get ,

p(s) $\int_0^\infty e^{-q(s)t}$ [erf $\sqrt{t}$ * erf $\sqrt{t}$] dt = $\frac{p(s)}{q(s)^3+q(s)^2}$

taking q(s)=5 in above equation, we get,

p(s) $\int_0^\infty$ e^{-5t} [erf $\sqrt{t}$ * erf $\sqrt{t}$] dt =$\frac{p(s)}{(5)^3+(5)^2}$

p(s) $\int_0^\infty$ e^{-5t} [erf $\sqrt{t}$ * erf $\sqrt{t}$] dt =$\frac{p(s)}{150}$

$\int_0^\infty$ e^{-5t} [erf $\sqrt{t}$ * erf $\sqrt{t}$] dt =$\frac{1}{150}$

VII. CONCLUSION:

In this paper, we have successful used the General Integral Transform of error function, to obtain the solution of Improper Integrals, which contain the Error Function. This show that General integral transform give the exact solution without any complete calculation work.

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